

MAGICAL Framework Describing the Nature of Students' Use of Representations

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Introduction

What is the nature of students' use of representations in a technological environment? That question raises great discussion along many psychological and mathematical lines. The specific interest here however is narrow. My intent is to describe a framework for analyzing students' actions on and uses of representations that students offer or encounter as they work in the presence of mathematics technology (e.g., computer algebra systems, dynamic construction environments, spreadsheets, graphing utilities). I will end with comments how this perspective relates to other literature.

Focus on *Shared Representations*

The focus of this paper is on representations of mathematical entities of importance in students' work with mathematics technology. Mathematics technology refers to electronic tools that assist in the learning and doing of mathematics. These tools include calculators, dynamic construction environments, spreadsheets, and computer algebra systems. I am focusing conscientiously on representations of concepts central to the intended and implemented curriculum in the form of text, teacher, interview, and group tasks while acknowledging there are many representations present in most any mathematical work or exchange.

To address students' use of external representations in the presence of mathematics technology, I think in terms of *shared representations*. They are potential representations of mathematical ideas and they are available to students and to others in the setting. Shared representations may be introduced by a student, the teacher, a curriculum developer, a researcher, or a software designer. A shared representation is a *potential* representation in that a particular student or other person may not see that shared representation as a representation of the mathematics that the student, teacher, researcher, curriculum developer, software designer, or other intended to represent. Examples of shared representations include textbook illustrations of a Cartesian graph of a function or an animated Geometer's Sketchpad file representing a family of isosceles triangles. There is no claim that shared representations necessarily are (or are not) manifestations of students' internal or external representations. Rather, these are potential representations that are present when students and others do mathematics and communicate mathematics in social settings. The very fact that two or more people may not see a shared representation as an instantiation of the same mathematics is part of what makes consideration of these representations essential to understanding learning and doing mathematics in classroom settings.

Shared representations need not be visually available. They include representations that are conveyed via words, writing, physical objects, body motions, or technology. These representations may be of many types, including Cartesian graphs, literal symbols, pictures, and geometric sketches. There are also *representations over time*, representations that do not exist in static forms. Examples include an arching movement of the arm to represent a parabola or the use of novel dynamic electronic environments to represent functions that

map real numbers to real numbers. The affordances of technology to bring novel and dynamic representations into a classroom and to offer opportunities to interact with these shared representations further underscores the need to discuss how students relate to shared representations.

What framework can I use to describe how students use and act on the shared representations they are given as well as how they relate to the shared representations they introduce? How can I use this framework to analyze data from multiple sources, including student interviews, small group work, and whole-class observation? Those are the questions addressed next in this paper.

Assumptions Underlying the Framework

The framework to capture students' uses of and actions on shared representations needs to have the following characteristics:¹

- 1) It allows multiple types of representations (e.g., table and graph of a function).
- 2) It allows multiple representations of the same type for two different mathematical things (e.g., two symbolic expressions representing different linear functions).
- 3) It includes, and in fact emerges from, literature involving symbolic (algebraic literal) representations.
- 4) It accounts for the use of multiple representations in algebraic or function settings yet also encompasses the use of multiple representations in other mathematical content settings (e.g., geometric figures, upper-level college-mathematics notation).
- 5) It allows for but does not demand (electronic) technology use.
- 6) It allows for associating representations with mathematical ideas and it allows for relating a mathematical idea with other mathematical ideas.
- 7) It allows for existing, emerging, established, and novel types of representations.
- 8) It allows for the reality of the classrooms in which we work but remain more globally relevant.

Consideration of multiple types of representations, as well as, consideration of multiple representations of one thing, leads to thinking about representational settings.

Representational Settings

Representational setting refers to a combination of the type of representation and the mathematical entity, or thing, represented. It is essential to note that the discussion here involves a type of representation and not a representation. [There is a correspondence between type of representation and register (Duval, 1999) and a correspondence between mathematical entity and equivalence class.] A two-letter code will carry information about the types of representations and types of mathematical entities represented. The first letter indicates representations of the same type (S) or of different types (D). The second letter denotes whether one (the same) (S) or several different (D) mathematical entities are represented. Thus, there are four possible codes for representational settings:

¹ This list may continue to emerge.

S-S

S-D

D-S

D-D.

For example, use of $f(x) = (x - 2)^2$ and $f(x) = x^2 - 4x + 4$ to represent the same function would be coded S-S. They are same type of representation (symbol literal algebraic) and they represent the same mathematical thing. In contrast, $f(x) = (x - 2)^2$ and $f(x) = (x - 3)^2$ would be coded S-D. They are same type of representation, but they represent different functions. [For completeness, the presence of only one representation is coded as “S-S” for same representation type of the same mathematical thing.]

A conjecture established early in my thinking about shared representations and seemingly supported in our early analysis of the data is, D-D settings will be left unresolved or will be resolved by introducing new structures in one of at least three ways. Using a common task (see Figure 1) as an example helps to illustrate how the resolution of a D-D setting may unfold in representationally different ways.

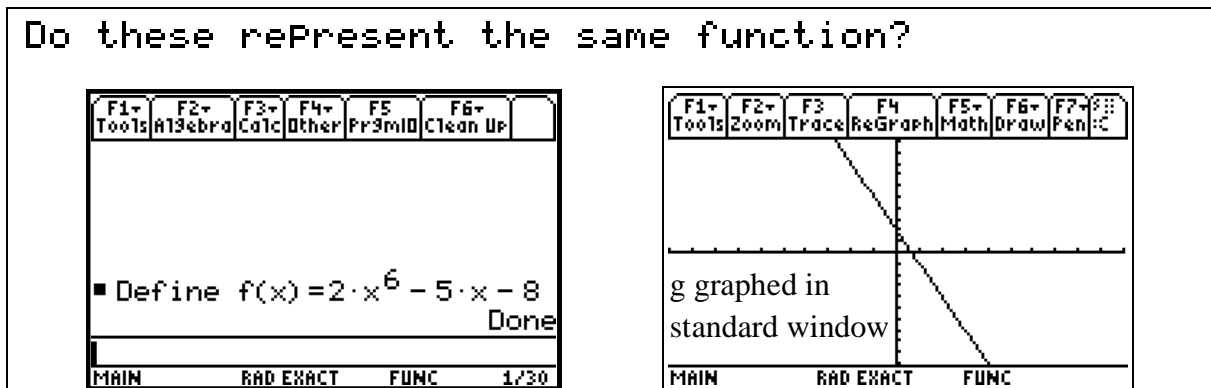


Figure 1. Task to exemplify resolutions of D-D representational setting

Suppose students have the task shown in Figure 1. Three ways in which students may attempt to resolve this D-D task are:

- 1) D-D / D-S / S-D combination: The initial setting is D-D, with different representation types (symbols and Cartesian graph) of two different functions (one a polynomial function of degree 6 and the other perhaps a linear function). Through a production act students create a second representation type for one of the two mathematical things, thus working in a D-S setting as they now have two different representation types (symbols and Cartesian graph) of the same function. The two different mathematical things (the two functions) are then discussed in terms of the common or same representation type.

Example: Student graphs f in a standard window (the D-S event), and then claims $f \neq g$ since the graphs look different (the S-D event).

- 2) D-D / D-S&D-S / S-D combination: The initial setting is D-D. Then, through a pair of production acts (each a D-S event), the students create a new representation in a third

representation type for each of the two mathematical things. The two mathematical things are then discussed in terms of this common (third) representation type (the S-D event).

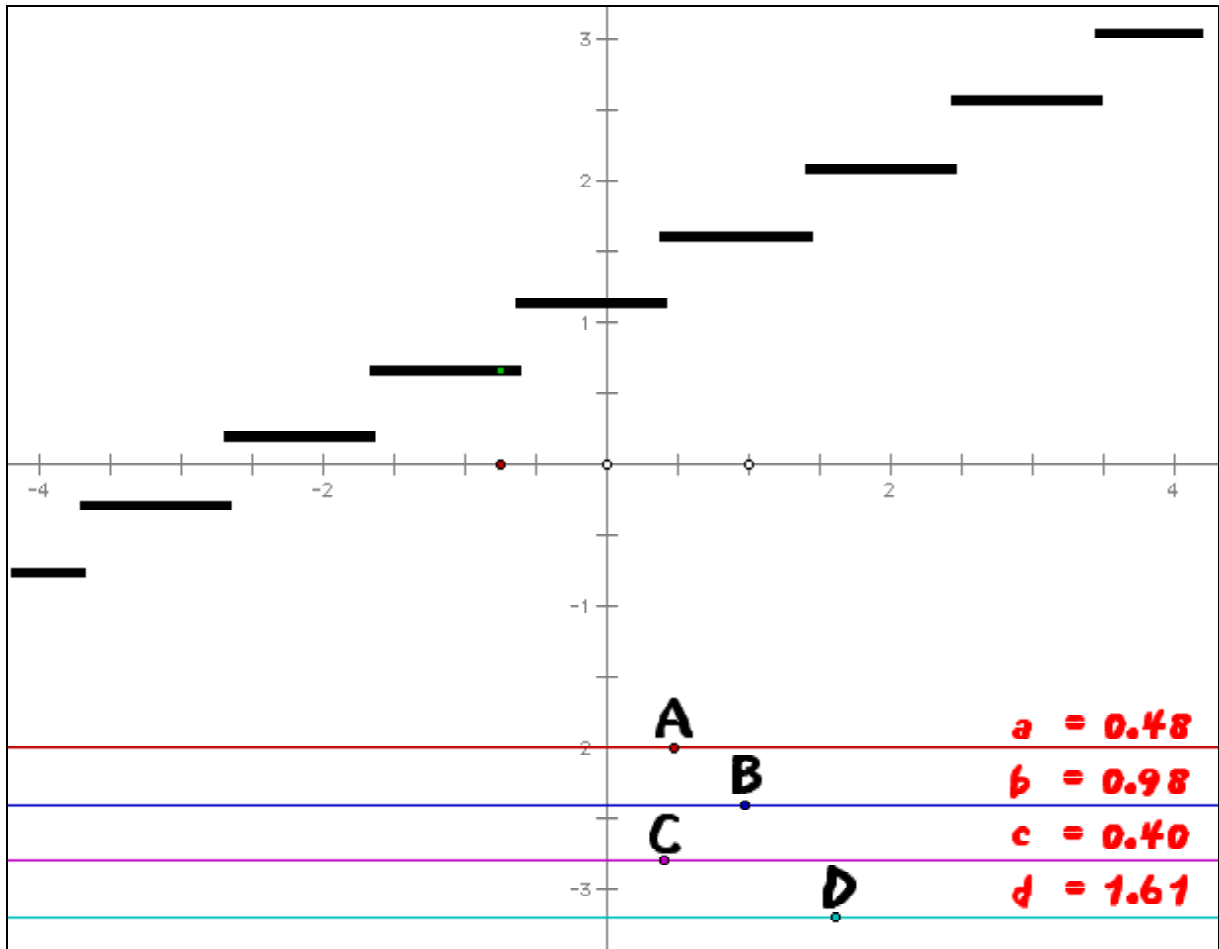
Example: Student uses auto table-graph connection to get a table of g (a D-S event) and generates a table of f (another D-S event) and compares the two tables (a S-D event) to claim $f \neq g$.

- 3) D-D / S-S & S-S combination: The initial setting is D-D. Each of the two different things still in its original representation type is interpreted in light of a common property (two S-S events). After the two interpretation acts, the two mathematics things are discussed in terms of this common property (with emphasis on the property and not directly on representations).

Example: Student notes vertical intercept of f is negative using "-8" in this symbolic form (an S-S event as it involves only the symbolic representation type of only the one mathematical function). Student also notes the vertical intercept of g is positive according to the graph (another S-S event but now with the Cartesian graph as the representation type and the g as the mathematical object). Students use the dissimilar intercept properties to claim $f \neq g$.

Despite the consistency of this perspective on resolving D-D episodes with the first portions of the data we analyzed, we eventually encountered multiple episodes in which D-D settings were not associated directly and explicitly with D-S, S-D, or S-S episodes. One example is from our March-28 small group observation. The students are working with a slider graph, our term for a dynamic representation of a family of functions. The students were using a slider graph that displayed a ceiling family function of the form $f(x) = d[b(x - c)] + d$; as they dragged the slider points at the bottom of the screen, the values of a , b , c , or d would change and the Cartesian graph would change appropriately to display a new member of the function family. [A rough picture of a slider graph is embedded in the excerpt that follows the next sentence.] The researcher asked the small group about the effects of changing the value of d and the students responded:

- R: So what do you think d does?
- S1: Causes it to go up if you move it right and down if you move it left.
- R: Okay.
- S2: When it goes left, it goes vertically down.
- S3: Moves it down. [S3 drags point D so that the value of d is about -1.61 as shown in the following picture]



All three students link different types of representations (slider and Cartesian graph) for different mathematical entities (parameter d and members of the ceiling function family, respectively). This would be a D-D event that was not resolved through D-S, S-D or S-S means.

D-D events that do are not neatly resolved into D-S, S-D, or S-S episodes, such as this ceiling function family example, seem to arise when learners are using dynamic representations. It seems that there may be something unique about these representational settings. Perhaps the dynamic nature makes more apparent to learners particular aspects of the mathematics involved, thus allowing the learners to work exclusively from internal representational acts of the D-S or S-D nature. It is also possible the mathematics contexts we used or the small group settings in which the events occurred contributed to the learners' lack of sharing D-S and S-D shared settings. Or, perhaps this is simply a matter of the students in the dynamic environment using an environment-favored type of representation – namely, movement – to represent two types of changes – such as change in parameter value and change in graph location during the small-group excerpt. If that is the case, a D-D event of this type is resolved through an appeal to a S-D event, with the same

(i.e., movement) type of representation representing different entities (i.e., two different types of changes).

The MAGICAL Categories

In addition to the representational setting, I describe the nature of the use of or action on the representation(s). The categories I have match the acronym, MAGICAL. Table 1 contains the MAGICAL code letters, the representation act abbreviated by each of those letters, and a description of the act. Appendix A contains examples in addition to the Table-1 descriptions of these categories.

Some of the MAGICAL categories require the setting to have only one representation type. Augmenting (Au), Manipulating (M), and Connecting (C) are actions done within one representation type, as illustrated in Figure 2. Others MAGICAL categories denote actions that require one representation type and a situation¹ or an abstract concept. This is the case with Interpret (I) and Ascribe (As), as indicated in Figure 3. The remaining MAGICAL categories require the presence of two types of representations in the setting. Figure 3 includes the situations for Generating (G) and Linking (L).

The relationships among representational settings and MAGICAL acts embodied in Figures 2 and 3 lead to the use of ordered triples to code representational uses or actions. Each triple is of the form, (*Same or Different representation type, Same or Different mathematical thing, MAuGICAsL category*). For example, S-D-C indicates a representational setting with the Same representation type of two or more Different mathematical things with the representations being Conected. An example of S-D-C is comparing the vertical intercepts in the Cartesian graphs of two different functions. As suggested by my use of Figures 2 and 3, some triples are not possible. For example, S-S-G is not a meaningful triple. It begins with "S" but Generating (G) requires moving between two Different types of representations.²

¹ "Situation" is used intentionally. I avoid "word" or "verbal" since it is not any word or anything verbalized that counts as a representation of a mathematical entity.

² Types of representations include: Cartesian graph, slider, slidergraph, dynamap, Dynagraph, table, numeric, symbolic, point. Deeper discussion related to types of representations appears later in this paper.

Table 1. The MAGICAL categories

<i>MAGICAL CODE LETTER AND ACT</i>	<i>DESCRIPTION OF THE CATEGORY</i>
M MANIPULATE	Change one representation of a mathematical thing to another representation of the same type. This includes standard symbolic manipulation.
AU AUGMENT	Make prominent something that is already in a representation. This changes the representation's visual appearance but does not change the essential parts of the representation and it does not change either the type of representation or the mathematical thing.
G GENERATE G2 GENERATE AGAIN BEYOND RECALL	This is the production or introduction of a representation of a particular representational type coming directly from a representation of another type. The new representation may come from an existing representation or from a situation embedded in a (seemingly) real-world context. The actual production may be done via technology, via learner or via teacher. ³ When the same representation is brought back without being regenerated, the event is not coded. However, G2 used for a repetition of the generation.
I INTERPRET	This involves giving meaning to a representation by interpreting it in terms of a situation or in terms of an abstract concept. This includes identifying a property of the concept within the representation.
C CONNECT/ COMPARE	Connect or compare two representations of the same type. They represent two different mathematical things. This naturally involves Interpret (I) but transcends I by involving two same-type representations of two mathematical things.
AS ASCRIBE AS2 ASCRIBE AGAIN	Produce the first representation of a situation or of an abstract concept. This includes the basic (initial) generation of a mathematical model. Ascribe may be used for other than the first representation of any type. This holds in situations where the creation of the representation happens when the “new” representation does not come directly from an existing representation.
L LINK	For one mathematical thing (equivalence class) a representation of one type is connected to a representation of another type. This link may focus on components of the mathematical thing, thus a linking passage may include a subpassage with a representation of the targeted component.

³ This list likely is not complete. It need not be complete at this point. The basic message is that the student does not have to be the creator and the technology has a potential role in a Generate act.

Types of Representations and What Is Represented

In thinking about types of representations and the role of these types in representational settings leads to asking what are possible types of representation. Some possibilities are obvious. A prime example of obvious types would be table, graph, and literal symbol as three types of representations used to express functions mapping real numbers to real numbers. However, meaningful ways to code events involving these representation types were not always immediately clear. Difficulty in determining codes also arose when working with less familiar shared representation types of function and of other mathematical ideas. Further thinking about the range of mathematical entities, how these entities can be represented, and the actions on those shared representations led to several key coding decisions.

What is Represented

There are many types of mathematical entities that can be represented. Some of these types are less obvious at the first pass. For example, two function graphs may coexist on the same axes. It is possible for the graph combination of the two co-present functions to be treated as one mathematical object. An example of this occurs when students have the graphs of f and g on the axes are using them to produce the graph of the sum, $f+g$. Students need to see the two graphs as representing "the addends" as well as separately representing the functions, f and g .

Distinguishing between MAGICAL Categories: Ascribe and Generate

Distinctions between certain categories may be more challenging than distinctions between other categories. If two representations, one of one type and one of another type, enter existence at the same time, their appearances are treated as two separate Ascribe acts. An example of this is when a student opens a book and sees the Cartesian graph of a function and the literal expression of the function on the same page. If a representation is already in the discussion and it is used to create a second representation of that type, the action may be either a Manipulate act or a Generate act. For example, one student moving from $h(x) = \lceil x \rceil - \lfloor x \rfloor$ on paper to $y1 = \text{ceiling}(x) - \text{floor}(x)$ on a TI-92 is coded as a manipulation. The student in this action substitutes $\text{ceiling}(x)$ for $\lceil x \rceil$ and $\text{floor}(x)$ for $\lfloor x \rfloor$. In contrast, suppose two students see a graph of a function and are asked to write a rule for it. One student writes $h(x) = \lceil x \rceil - \lfloor x \rfloor$ on the classroom whiteboard while a second student simultaneously enters $y1 = \text{ceiling}(x) - \text{floor}(x)$ into a TI-92 attached to the classroom display. The coding of both rule-production events would be Generate.

Replication of Acts

Repeats of previously acknowledged representation acts are not coded. For example, a student picking up the student's previously produced paper-and-pencil graph is not coded. Similarly, returning to the GRAPH screen to see again a previously created calculator graph is not coded. Hitting REGRAPH on a calculator to produce the same graph

of a function is not coded as a Generation task. (However, some REGRAPH events may be coded. A good example is when the researcher suggested students use REGRAPH to reproduce the graphs of two functions in order to see what points of one graph were also points of the second graph.) However, the second production of the same type of graph may be coded as a G2 if that second production is independent of a previous production. An example would be a student sketching on paper the graph of a function from its rule and then later re-producing a perceptibly identical sketch on another spot on the paper without reference of the graph's previous existence. We allow the possibility that distinction between G and G2 codings may become irrelevant in some analyses of the data.

When the creation of a representation of a new type is based on an existing shared representation⁴, the action is Generate. Ascribe is used in cases when there is no existing shared representation of the mathematical entity. S-S-As2 denotes the replication of an Ascribe act. This typically occurs when a representation of an existing type for that mathematical object is created via a means different from that used to create the previously created representation of that type. For example, if a student is asked what the Pythagorean theorem means and the student produces $\triangle ABC$ with a right angle at C on a computer screen and then sketches a seemingly similar $\triangle ABC$ with a right angle at C on paper, the two productions of the triangle would be coded S-S-AS and S-S-As2, respectively.

Representations of functions

There was a perhaps surprising need to distinguish among the actions involving representations of $R \rightarrow R$ functions. For example, given $f(x) = x^2$, one may sketch either of the drawings in Figure 4. The act of creating the picture in Figure 4(a) is coded D-S-G. The Different representations are the symbolic form into a picture. These represent the Same mathematical entity. The picture was Generated from the symbolic form. The act of creating the graph in Figure 4(b) is also coded D-S-G. The difference is that the representation in this case is a Cartesian graph rather than a picture.

⁴ The term “shared representation” is used intentionally here. There may be internal representations involved but these are not readily accessible for consideration as shared representations. I would not argue, for example, that an Ascribed representation does not come from an existing *internal* representation of some type.

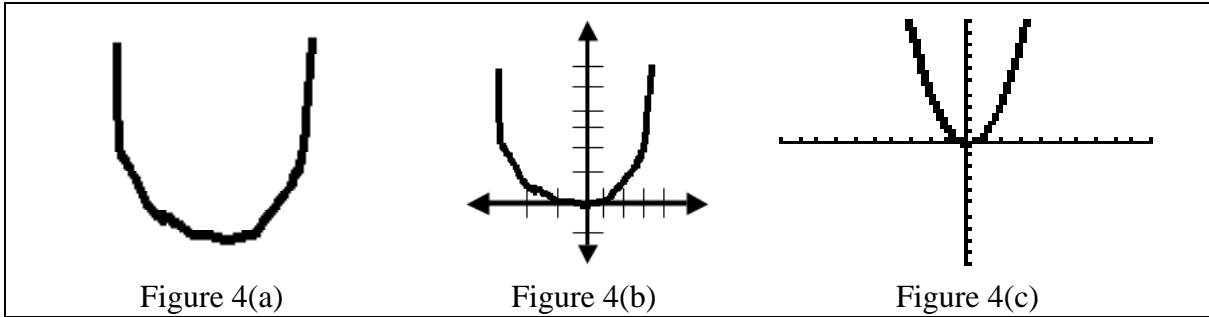


Figure 4. Sketches and CAS-generated graph given $f(x) = x^2$

The distinction noted for the two images in Figure 4 also has implications for interpreting the learners' subsequent work with shared representations. For example, assume the next representational act was the creation of a graph, Figure 4(c), for $f(x) = x^2$ with a computer algebra system. If this follows Figure 4(a), it is coded D-S-G for the generation of the first Cartesian graph from the symbolic form. If the CAS work follows the Figure 4(b) event, it is a replication (albeit by electronic tool) of the work already done to create the Figure 4(b) image and it therefore is not coded.

Numerical representations of numbers

Further thinking about types of representations for and beyond \mathfrak{R} - \mathfrak{R} functions is necessary. For example, the tables used to represent the \mathfrak{R} - \mathfrak{R} functions are considered by other mathematics educators to be numeric representations.⁵ There also are numeric representation types used when other mathematical entities are considered. For a point considered in the plane or in space, a numerical form may be the coordinate form of the point. For a number, perhaps ironically, thinking about numeric representations requires some refinement.

Consider the numbers commonly represented as 7, $\frac{1}{2}$, and 1.5. There are many ways to represent these values – a fact that underlies much of both the power and the complexity of elementary school mathematics. Table 2 contains several different ways of representing these three numbers.

⁵ I prefer to limit tables to those arrangements with input values in order and by constant increment in one column (or row) and the corresponding output values in another column (or row). The term "values" here is used to mean numeric values for \mathfrak{R} - \mathfrak{R} functions but may be extended to represent other orderly arrangements. When the input values lack a mathematically ordered structure, the 'table' is a chart. This chart/table distinction has been useful in communicating with students about functions and in helping them to distinguish between a chart of data values they record and the table of a function model of the relationship in those data.

Table 2. Varieties of numerical representations

Variety of Numerical Representation	7	$\frac{1}{2}$	1.5
Numeral (includes integer, fraction in simplified form, mixed numeral in simplified form)	7 7.0	$\frac{1}{2}$ 0.5	1.5 $1\frac{1}{2}$ $\frac{3}{2}$
Indicated numeric (expression without condition; typically a calculation or a special function symbol used)	$\lceil 7 \rceil$ $\sqrt{49}$ $\lfloor 7 \rfloor$ $\lceil 6.7249 \rceil$ $\sqrt{\left(\frac{4\pi + 3\pi}{\pi}\right)^2}$	$\frac{1}{4} + \frac{1}{4}$ $2 \div 4$ $\sin(30^\circ)$	0.5 $\times 3$ $\frac{1}{2} + \frac{1}{2} + \frac{1}{2}$ $ -1.5 $
Conditional numeric (contains condition as a separate or embedded component; may involve "letters")	TI-92 combination of Define $h(x) = \text{floor}(x)$ followed by $h(7)$ $x x^2 = 49, \quad x = x $	$x 2x - 1 = 0, \quad x \in \mathfrak{R}$ $h(2) \quad \text{if} \quad h(x) = x^{-1}$	$-x \lfloor 2x \rfloor = \lceil 2x \rceil$ and $x \in (-2, -1)$ $x \frac{1}{x-1.5} \notin \mathfrak{R}$ but $x \in \mathfrak{R}$

As suggested by the title of Table 2, I consider Numeral, Indicated numeric, and Conditional numeric *varieties* of numerical representation types. I chose to use this labeling system for the following reasons:

- 1) The varieties arise in CAS work. Distinctions among these varieties may have a relationship to different MAGICAL actions or to different technology uses. Since the MAGICAL categories should apply to students' uses of representations in technological settings, I needed both to address how to code episodes involving expressions like those in Table 2 and to have a structure in which meaningful relationships could arise. Thus, it helped to identify – and perhaps to name – these varieties of numeric representations.
- 2) Knowing the breadth of the numeric representation type was important. I did consider treating each of the varieties as a separate representation type. However, this created an excessively fine-grain view that seemed to cloud rather than illuminate important issues. For instance, consider the following shared representation events and how they would be coded if the varieties were treated as separate representation types:

a. $\frac{1}{2} + \frac{1}{4} + \frac{1}{4} \rightarrow \frac{1}{2} + \frac{1}{2}$

If the numeric varieties were different representation types, this would be coded S-S-M at the global level. Both would be the Same type of representation (indicated numeric) and both represent the Same mathematical entity (the sum). The MAGICAL action for this S-S setting would be Manipulation.

At the level of considering $1/4 + 1/4$ becoming $1/2$, if the numeric varieties were different representation types, this episode would also be coded D-S-G. $1/4 + 1/4$ (indicated numeric) and $1/2$ (numeral) would be Different types of representations that represent the Same mathematical thing. Changing the indicated numeric to the numeral would be a Generate action.

b. $\frac{1}{2} + \frac{1}{2} \rightarrow 1$

If the numeric varieties were different representation types, this would be coded as D-S-G. It would be a Generation of a Different type of representation (numeral from indicated numeric).

Thinking of “generating” with numbers to include events such as the action that yields 1 from $1/2 + 1/2$ had a meaning that deviated from the richness of the intended meaning and use of Generate. It also seemed to be in conflict with terminology in the field. Actions such as that which takes $1/2 + 1/2$ into 1 seemed to match better how “Manipulate” is typically used in the field.

- 3) Using varieties seemed more in line with how we treat symbolic representations. Consider the following symbolic forms at the end manipulations involving them; assume $abc \neq 0$:

a. $(a+b)(a+b)-(a+b)(a-b) -b^2$	b. $2((a^{-1/2}b^{3/2}c^9)/(a^{-1}bc^9))^2$	c. $ab+ab$
$a^2+2ab-b^2-a^2+b^2-b^2$	$2(a^{1/2}b^{1/2})^2$	$1ab+1ab$
$2ab$	$2ab$	$2ab$

We tend to consider all of these representations as symbolic forms and all of the actions as manipulations. Yet, some of the representations are more difficult to understand. Some of manipulations seem to be more or less difficult than other manipulations. The range of complexity in symbolic representations seems to be similar to the corresponding difficulty in interpretation of the different varieties of numeric representations. The difficulty in manipulation of these symbolic forms seems to reflect the difficulty that we have with changes from one variety of numeric representation to another variety of numeric representation, such as the following:

a. $\frac{1}{4} + \frac{1}{4}$	b. $x 2x - 1 = 0, x \in \mathfrak{R}$	c. $h(2)$ if $h(x) = x^{-1}$
$\frac{1+1}{4}$	$x 2x = 1, x \in \mathfrak{R}$	2^{-1}
$\frac{2}{4}$	$x x = \frac{1}{2}, x \in \mathfrak{R}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	

Potential for Nesting Representational Events

It is also essential to note that portions of representational work may be part of two or more representational episodes and settings. To see how this may happen, it helps to think about some ideas of mathematical interest. In the CAS-IM project⁶, I care about functions as well as families of functions. Ordered pair is an idea needed for function; parameter and function are ideas needed for families of function. These embedded needs lead to the overlap of episodes, as represented by the overlapping brackets in Figure 5. These brackets show how ordered pair is intrinsic to definition of function and so a shared representation of function likely has embedded in it representation(s) of ordered pairs. A parameter is intrinsic to definition of families of functions so a shared representation of a family of functions possibly has embedded in it representations of parameter. A note of caution is needed. The embedded brackets must come from the actual coding; they do not arise simply because the embedding is possible. To have the overlapping and embedded

⁶ The CAS-Intensive Mathematics project is a curriculum development and educational research funded by the National Science Foundation Grant No. TPE 96-18029 to The Pennsylvania State University with a major subcontract to The University of Iowa.

categories requires that both the super-concept and sub-concept are part of the conversation about the shared representation(s).

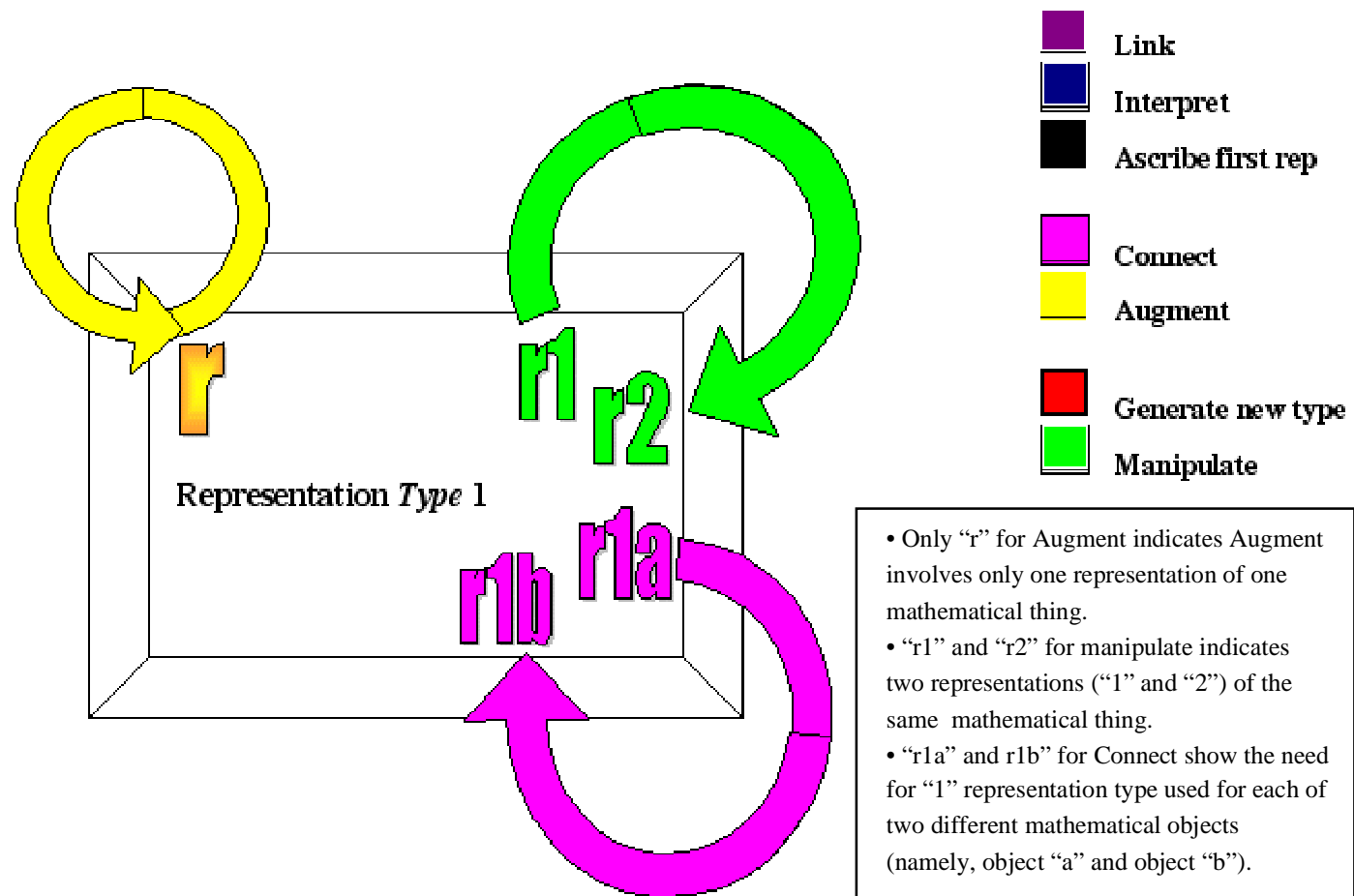


Figure 2. MAGICAL Categories Within One Representation Type

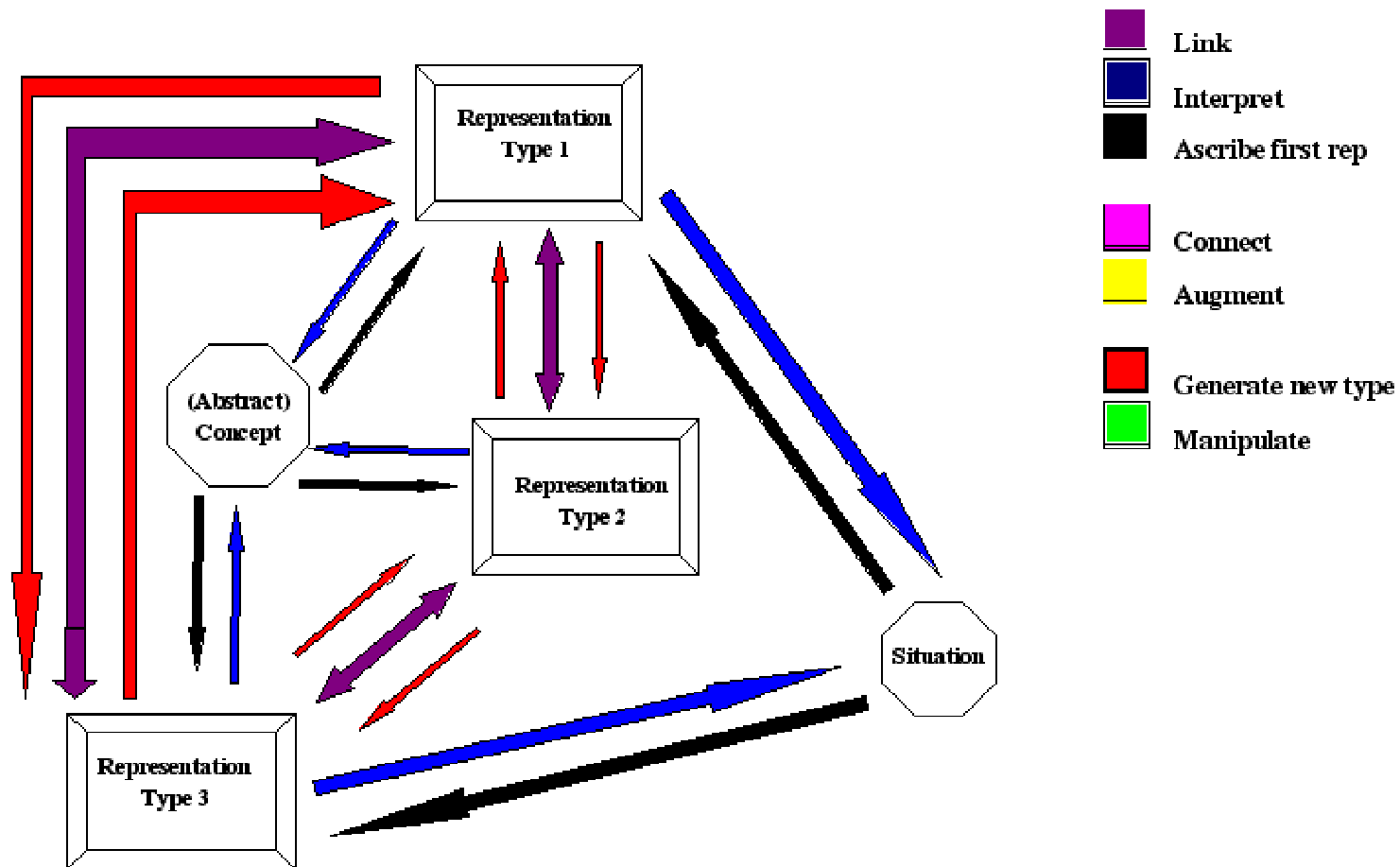


Figure 3. MAGICAL Categories Across Representation Types

(There certainly are more than three representation types, of course, but I am confined by the 2D limits of this page.)

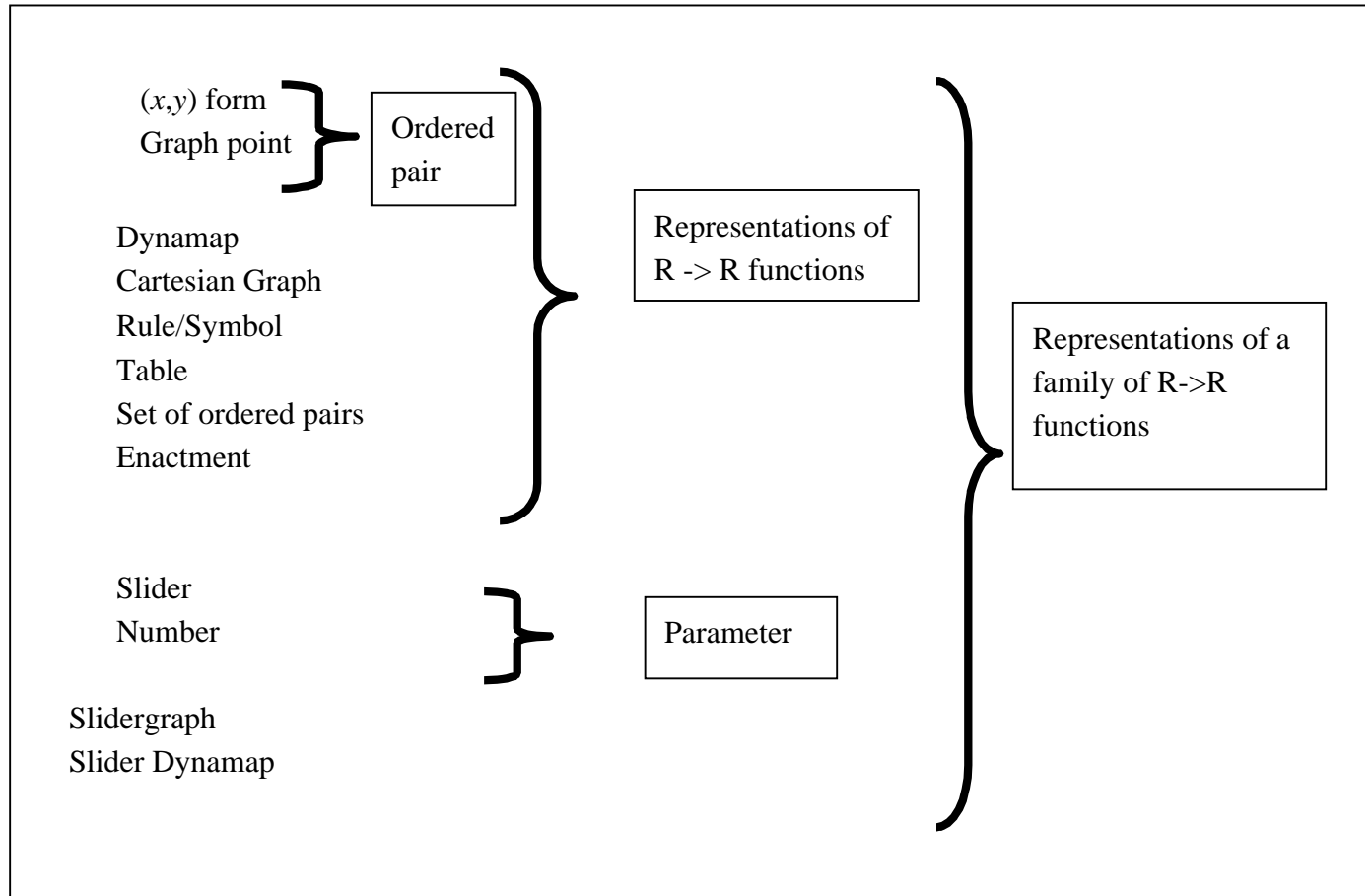


Figure 5. Overlapping nature of representational episodes arising from the nested nature of mathematical things of interest. (List is representative rather than comprehensive)

I can also group the MAGICAL categories by the influence of technology. More about this relationship between representation and technology will appear after a brief discussion of technology features and affordances.

Technology Features and Affordances

Interest in shared representations is partially an attempt to explore how affordances of technology relate to students' uses of and actions on representations. There are several *features of technology* that merit consideration as to how they may be viewed from the MAGICAL framework.

I do not consider feature of technology itself to be either a representation or an act with a representation. Certain features however seem to relate closely to particular components of the framework. One example is the TRACE feature in many graphing utilities. This feature seems to be related to Augmenting a representation, as when the TRACE feature is activated and a dot appears to make more prominent a point that (the

user acknowledges) was already on the Cartesian graph. At the same time, the TRACE feature often prints the coordinates of this point on the screen. These coordinates are a numerical representation of the point for which the dot is a geometric representation. Using the TRACE feature in this case also is a Generating act in that it produces the numerical-coordinate representation for the point already represented graphically as the dot. Figure 6 summarizes the TRACE feature relationships to these MAGICAL categories.

TRACE:	Au – Add points already in Cartesian graph. G – Produce coordinates for point.
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Figure 6. TRACE feature (in a graphing utility) relates to the MAGICAL categories of Augment and Generate.

A more complex insight comes from looking at the DRAG feature and how it affords various MAGICAL representation acts. DRAG relates to several different MAGICAL scenarios depending in part on when and how as well as in what software the feature is invoked. Figure 7 conveys several ways in which DRAG corresponds to representational actions. It is important to note that DRAG may occur with very different levels of intensity of mathematical thinking about or with the representations.

These variations on MAGICAL categories for a single feature imply several other important points about the ordered triples. First, the command, motion, or other technology feature should not be used as a determining factor for a coding triple.⁸ For instance, dragging as a feature associates with several MAGICAL categories. Further, the type of technology and the type of representation used also are not definitive indicators of the MAGICAL codes. The samples in Figure 7 are examples of this inability to use representation type or tool type to determine uniquely a MAGICAL code. The slidergraphs in the DRAG samples in Figure 7 are involved in Augment as well as in Link. Also, dynamaps and slidergraphs can both be created with Cabri Geometry or with The Geometer's Sketchpad. The correspondence of dynamaps or slidergraphs with Connect, Augment, and Link in Figure 7 indicate clearly that the geometry tools – like the use of their drag feature – do not uniquely define any one MAGICAL code.

⁸ I also ponder how the MAGICAL categories may help me to explain why CAS does not have the same effects or ease of use or the same ease of activity generation as do dynamic construction environments. For example, CAS embodies a type of expertise in Manipulating as a black box but the Geometer's Sketchpad or Cabri requires more thoughtful action on the user's part to have a Manipulation action. Augmenting may be harder with CAS than with Sketchpad or Cabri. In addition, CAS ease for Linking via linked multiple representations may underscore the often seemingly excessive use of graphical symbols over literal symbols in many CAS papers, curriculum materials and research reports (See Zbiek [In press] for additional comments on the issue of graphical form dominance).

M – Drag in Theorist ⁹ to factor.	<- These need not involve deep meaning.
G – Drag in Fathom ¹⁰ to create a scatter plot from data collection.	Critical mathematical work can be done by the technology.
C – Drag vertex point to change figure representing one triangle into a figure representing another triangle ¹¹ .	<- These involve changing representations to underscore meaning or features.
Au – Drag x and see $(x, f(x))$ move in a Cartesian slidergraph. ¹²	Technology can help.
I – This category of activity cannot be done.	<- These involve giving or expressing meaning.
As – This category of activity cannot be done.	
L – Compare ordered pairs and motion of points as one drags input point of a dynamap.	Technology may facilitate visualization but it does not help in the critical act of comparing.

Figure 7. DRAG feature may have one of several relationships to the MAGICAL categories.

Data to Code

Our research group used the MAGICAL codes initially with individual student interviews. We also used the MAGICAL categories to analyze portions of the small

⁹ Theorist refers to the symbol manipulation software MathView.

¹⁰ Fathom is the dynamic statistics software recently released by Key Curriculum Press/Key College Press.

¹¹ This example refers to dragging in a dynamic construction environment.

¹² This is done in the CAS-IM curriculum and research work. We use *slidergraphs* to represent families of functions. Dragging points along horizontal lines (“sliders”) at the bottom of the screen allows the user to change the values of the family parameters and thus denote over time various members of a family of functions. A Cartesian graph of the family member matching the parameter choices at the given moment appears on the screen and is updated automatically.

group work and classroom observations. We similarly used the categories to analyze portions of the written textbook, the answer key, the interview protocol, the small group task, and the teacher interviews. The coding of curriculum materials focused on the content most immediate in both time and content to the time and content of the corresponding interviews, small group work, and classroom observations.

In any of these data arenas, we coded passages that involve not only single students but passages that include the work of several students and/or adults. Interviewer, researcher, teacher, collaborator, and other adult elements (e.g., their questions, statements, tech moves) are included in passages. If students did not follow up on a teacher element, that teacher element was coded as a separate passage.

In our work, MAGICAL categories have been good tools for careful, collaborative fine interpretation of student work with representations in the presence of computing tools. The MAGICAL perspective, however, is not intended to be a tool for rapidly coding long passages.

Origin of MAGICAL Ideas

Existing expository and empirical literature naturally influenced the MAGICAL framework. “Multiple linked representations” for functions and the question of the role of symbolic manipulation in CAS settings were key starting points. **L**inking and **M**anipulating thus arose. Thinking about symbolic manipulation as a means for changing forms but not types of representation led to using types of representations as an essential part of the framework. Work with representations, with early exposure to the “graph as picture” misconception (Kerslake, 1977), brought out the idea of **I**nterpreting. Thinking about the connection and representation process standards (NCTM, 2000) raised the issue that there is a **L**inking across different types of representations but there is also a **C**onnecting of representations of the same type. **C**onnecting and **L**inking were distinguished also because of how differently software tends to support the thinking needed for these representational acts. Modeling literature and the creation of a mathematical model inspired the need for **G**enerating. Creating examples that were not previously part of a mathematical discourse suggested a need for introducing new mathematical objects and thus led to the distinction between **A**scribing a representation of a new mathematical object (as opposed to **G**enerating a representation of a different type for a mathematical object already shared through another representation type). Work with tracing features of graphing calculators and other features of the technology as well as the communicative function of representation use contributed to the need to include **A**ugmenting in the framework.

Relation to Frameworks and Models in the Literature

The resulting MAGICAL framework is consistent with several other writers' work, and particularly with literature involving algebra, function, and technology. For example, Kieran (1996, 2001) discusses several types of algebraic activity. Two of the three categories seem to have a direct parallel in the MAGICAL framework. Her generating and transformation match Generating and Manipulating, respectively. Aspects of Kieran's meta category relate in various ways to the categories described here. For example, noting structure relates to both Linking and Connecting. Modeling relates to Ascribing and Interpreting. Kieran's justifying and proving within the meta category typically involve multiple MAGICAL actions. One may seek other comparisons between Kieran's algebraic activities and the MAGICAL representational activities. In comparison to Kieran's discussion of algebraic activity, the MAGICAL framework is both a finer grain view and a perspective that may be applied beyond as well as within algebraic arenas.

The MAGICAL framework also extends several other perspectives. It could be used to elaborate on Schneider's (2000) discussion of representation, operation, and interpretation with CAS. MAGICAL thinking is also consistent with the reification model used by O'Callaghan (1998) to describe student understanding within a course using Computer-Intensive Algebra. Modeling would be Ascribe; Interpreting is Interpret; Translating involves Generating and Linking; Reifying would be indicative of one who can move within and among the MAGICAL aspects with several types of representation. In addition, students' fluency with some parts of the MAGICAL categories may explain what Pierce (2001) describes as algebraic insight.

The MAGICAL framework transcends all of these in that it transcends algebra, function, and CAS. These MAGICAL categories of actions on shared representations should apply to any mathematical concept and should be appropriate with any mathematical tool.

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APPENDIX A.

Framework Codes with Descriptions and Examples

This reiterates and elaborates on ideas from the MAGICAL framework paper. It includes examples and particular comments, in a form that parallels the TECH USE codes for types and purposes of technology.

<i>Code</i>	<i>Description</i>	<i>Examples</i>
M Manipulate	Change one representation of a mathematical thing to another representation of the same type. This includes standard symbolic manipulation.	Solve equation with CAS. Change window for graph by ZOOM or by WINDOW. <i>Key idea: This changes the form and the information given by the representation.</i>
Au Augment	Make prominent something that is already in a representation. This changes the representation visually but does not change the essential parts of the representation and it does not change either the type of representation or the mathematical thing.	TRACE a graph. Add color, line format, or shading to a geometry figure (as in make the hypotenuse of right triangle red or make the altitudes dotted) <i>Key idea: This adds no new information but makes existing information more apparent.</i>

Two representations of the same mathematical thing: If new information is added, the code is Manipulate.

If there's no new information added, the code is Augment.

G (G2) Generate	This is the production or introduction of the first representation of that type. The new representation may come from an existing representation or from a situation embedded in a (seemingly) real-world context. The actual production may be done via technology, via learner or via teacher. ¹³	Having the function rule in the CAS, produce the first graph of the function. Later creating the graph on paper from the rule would be coded G2. Coming back to the graph without regenerating it would not be coded. Examples of the uncoded events include: regraphing (e.g., F4 on TI-92) the same graph, later picking up the CAS and seeing the graph later, or returning to the graph window after being out of it. <i>Key idea: This requires movement from one representation TYPE to another representation TYPE.</i>
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Generate is different from Manipulate based on the types of representations involved:

Manipulate moves from one representation of the ONE TYPE to another representation of the SAME TYPE.

Generate moves from one representation of ONE TYPE to a representation of ANOTHER TYPE.

I Interpret	This involves giving meaning to a representation by interpreting it in terms of a situation or in terms of an abstract concept. This includes identifying a property of the concept within the representation.	Looking at the constant term of a polynomial function expression, identify the vertical intercept. Using a graph, identify the monotonic nature of a function. <i>Note: This requires mathematical meaning rather than visual description.</i>
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¹³ This list likely is not complete. It need not be complete at this point. The basic message is that the student is not the only creator and the technology has a role in this.

C Connect/ Compare	Connect or compare two representations of the same type. They represent two different mathematical things. This naturally involves I but transcends I by involving two same-type representations of two mathematical things.	Noting floor rounds down and ceiling rounds up while looking at the graphs of both functions on the same graph. Looking at rules $f(x) = x$ and $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$, note that the identity function and the absolute value function have the same output for positive numbers. <i>Key idea: The use of two representations of the same type allows one to compare (find similarities or differences among) two mathematical ideas.</i>
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Connect is different from Interpret in that Connect has two different mathematical ideas involved while Interpret involves only one mathematical idea.

As Ascribe	Produce the first representation of a situation or of an abstract concept. This includes the basic generation of a mathematical model.	Construct a rectangle to represent a field for a maximum-area/minimum-perimeter fencing-the-field problem. Fit a function to data points. Having previously fitted a quadratic function to data points now fit a quartic function to those data points. <i>Key idea: The situation is mathematized in a new way. This includes the creation of a (new) model.</i>
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Generate is moving from representation to representation. Ascribe is moving from idea to representation.

L Link	For one mathematical thing (equivalence class) a representation of one type is connected to a representation of another type. This may involve components of the mathematical thing, thus a linking passage may include a subpassage with a representation of the targeted component.	The constant term of a polynomial function expression is related to the vertical intercept. The constant terms in the linear factors in the completely factored form of the denominator of a rational function are related to the vertical asymptotes. <i>Key idea: The focus is on a mathematical feature of a mathematical entity that is represented in two different forms.</i>
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If the representations are of the SAME FORM, the action is Connecting rather than Linking.