# MAGICAL Framework Describing the Nature of Students' Use of Representations

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## Introduction

What is the nature of students' use of representations in a technological environment? That question raises great discussion along many psychological and mathematical lines. The specific interest here however is narrow. My intent is to describe a framework for analyzing students' actions on and uses of representations that students offer or encounter as they work in the presence of mathematics technology (e.g., computer algebra systems, dynamic construction environments, spreadsheets, graphing utilities). I will end with comments how this perspective relates to other literature.

## Focus on Representation

The focus of this paper is on representations of mathematical entities of importance in students' work with mathematics technology. This acknowledges that there are many representations present in most any mathematical work or exchange but I am conscientiously focusing on representations of concepts central to the intended and implemented curriculum of the text, teacher, interview, and group task.

To address students' use of representations in the presence of mathematics technology, I think in terms of *shared representations*. These are potential representations of mathematical ideas and they are available to students and to others in the setting. They may be introduced by a student, the teacher, a curriculum developer, a researcher, or a software designer. A shared representation is a *potential* representation in that a particular student or other person may not see that shared representation as a representation of the mathematics that the student, teacher, researcher, curriculum developer, software designer, or other intended to represent. Examples of shared representations include textbook illustrations of a Cartesian graph of a function and an animated Geometer's Sketchpad file representing a family of isosceles triangles.

There is no claim that shared representations necessarily are (or are not) manifestations of students' internal or external representations. Rather, these are potential representations that are present when students and others do mathematics and communicate mathematics in social settings. The very fact that two or more people may not see a shared representation as an instantiation of the same mathematics is part of what makes consideration of these representations essential to understanding learning and doing mathematics in classroom settings.

Shared representations need not be visually available. They include representations that are conveyed via words, writing, physical objects, body motions, or technology. These representations may be of many types, including Cartesian graphs, literal symbols, pictures, and geometric sketches. There are also *representations over time*, representations that do not exist in static forms. Examples include an arching movement of the arm to represent a parabola or the use of novel dynamic electronic environments to represent functions that map real numbers to real numbers. The affordances of technology to bring novel and dynamic representations into a classroom and to offer opportunities to interact with these shared representations further underscore the need to discuss how students relate to shared representations.

What framework can I use to describe how students use and act on the shared representations they are given as well as how they relate to the shared representations they introduce? How can I use this framework to analyze data from multiple sources, including student interviews, small group work, and whole-class observation? Those are the questions addressed next in this paper.

## Assumptions Underlying the Framework

The framework to capture students' uses of and actions on shared representations needs to have the following characteristics:[1]

1) It allows multiple types of representations (e.g., table and graph of a function).
2) It allows multiple representations of the same type for two different mathematical things (e.g., two symbolic expressions representing different linear functions).
3) It includes, and in fact emerges from, literature involving symbolic (algebraic literal) representations.
4) It accounts for the use of multiple representations in algebraic or function settings yet also encompasses the use of multiple representations in other mathematical content settings (e.g., geometric figures, upper-level college mathematics).
5) It allows for but does not demand (electronic) technology use.
6) It allows for associating representations with mathematical ideas and it allows for relating a mathematical idea with other mathematical ideas.
7) It allows for existing, emerging, established, and novel types of representations.

I base these assumptions on the reality of the classrooms in which I work but suspect they are more globally relevant. Consideration of multiple <u>types</u> of representations, as well as, consideration of multiple representations of one thing, leads to thinking about representational settings.

## Representational Settings

*Representational setting* refers to a combination of the type of representation and the mathematical thing represented. It is essential to note that the discussion here involves <u>a type of</u> representation and not <u>a</u> representation. [There is some degree of correspondence between type of representation and register (based on discussion in Laborde, Straesser, & Hollebrands, [in press]) and between mathematical thing and equivalence class.] A two-letter code will carry information about the types of representations and types of mathematical entities represented. The first letter indicates representations of the <u>s</u>ame type (S) or of <u>d</u>ifferent types (D). The second letter denotes whether one (the <u>s</u>ame) (S) or several <u>d</u>ifferent (D) mathematical things are represented. Thus, there are four possible codes for representational settings:

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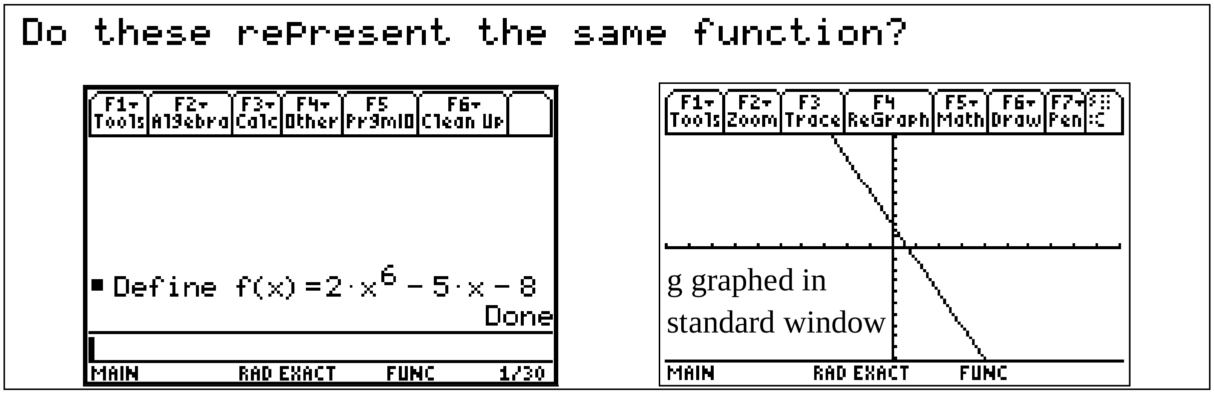
[1] This list may continue to emerge.

S-S S-D D-S D-D.

For example, use of $f(x)=(x-2)^2$ and $f(x)=x^2-4x+4$ to represent the same function would be coded S-S. They are same type of representation (symbol literal algebraic) and they represent the same mathematical thing. In contrast, $f(x)=(x-2)^2$ and $f(x)=(x-3)^2$ would be coded S-D. They are same type of representation, but they represent different functions. [For completeness, the presence of only one representation is coded as "S-S" for same representation type of the same mathematical thing.]

A conjecture established early in my thinking about shared representations and seemingly supported in our early analysis of the data is, D-D settings will be left unresolved or will be resolved by introducing new structures in one of at least three ways. Using a common task (see Figure 1) as an example helps to illustrate how the resolution of a D-D setting may unfold in representationally different ways.



Do these represent the same function?

■ Define $f(x)=2\cdot x^6-5\cdot x-8$
Done

Figure 1. Task to exemplify resolutions of D-D representational setting

Suppose students have the task shown in Figure 1. Three ways in which students may attempt to resolve this D-D task are:

1) D-D / D-S / S-D combination: The initial setting is D-D, with different representation types (symbols and Cartesian graph) of two different functions (one a polynomial function of degree 6 and the other perhaps a linear function). Through a production act students create a second representation type for one of the two mathematical things, thus working in a D-S setting as they now have two different representation types (symbols and Cartesian graph) of the same function. The two different mathematical things (the two functions) are then discussed in terms of the common or same representation type.
   *Example*: Student graphs *f* in a standard window (the D-S event), and then claims $f\neq g$ since the graphs look different (the S-D event).
2) D-D / D-S&D-S / S-D combination: The initial setting is D-D. Through a pair of production acts (each a D-S event), the students create a new representation in a third

representation type for each of the two mathematical things. The two mathematical things are then discussed in terms of this common (third) representation type (the S-D event).

*Example*: Student uses auto table-graph connection to get a table of *g* (a D-S event) and generates a table of *f* (another D-S event) and compares the two tables (a S-D event) to claim $f \neq g$.

3) D-D / S-S & S-S combination: The initial setting is D-D. Each of the two different things still in their original representation types is interpreted in light of a common property (two S-S events). After the two interpretation acts, the two mathematics things are discussed in terms of this common property (with emphasis on the property and not directly on representations).

*Example*: Student notes vertical intercept of *f* is negative using "-8" in this symbolic form (an S-S event as it involves only the symbolic representation type of only the one mathematical function). Student also notes the vertical intercept of *g* is positive according to the graph (another S-S event but now with the Cartesian graph as the representation type and the g as the mathematical object). Students use the dissimilar intercept properties to claim $f \neq g$.

## The MAGICAL Categories

In addition to the representational setting, I describe the nature of the use of or action on the representation(s). The categories I have match the acronym, MAGICAL. Table 1 contains the MAGICAL code letters, the representation act abbreviated by each of those letters, and a description of the act. Appendix A contains examples in addition to the Table 1 descriptions of these categories.

Some of the MAGICAL categories require the setting to have only one representation type. Augmenting (Au), Manipulating (M), and Connecting (C) are actions done within one representation type, as illustrated in Figure 2. Others MAGICAL categories denote actions that require one representation type and a situation or an abstract concept. This is the case with Interpret (I) and Ascribe (As), as indicated in Figure 3. The remaining MAGICAL categories require the presence of two types of representations in the setting. Figure 3 includes the situations for Generating (G) and Linking (L).

The relationships among representational setting and MAGICAL act embodied in Figures 2 and 3 lead to the use of ordered triples to code representational uses or actions. Each triple is of the form (*Same or Different representation type, Same or Different mathematical thing, MAuGICAsL category*). For example, S-D-C indicates a representational setting with the Same representation type of two or more Different mathematical things with the representations being Connected. An example S-D-C would be comparing the vertical intercepts in the Cartesian graphs of two different functions. As suggested by my use of Figures 2 and 3, some triples are not possible. For example, S-S-G is not a meaningful triple. It begins with "S" but Generating (G) requires moving between two Different types of representations. Thus triples ending in "G" must start with "D".

### Potential for Nesting Representational Events

It is also essential to note that portions of representational work may be part of two or more representational episodes and settings. To see how this may happen, it helps to think about some ideas of mathematical interest. In the CAS-IM project[2], I care about functions as well as families of functions. Ordered pair is an idea needed for function; parameter and function are ideas needed for families of function. These embedded needs lead to the overlap of episodes, as represented by the overlapping brackets in Figure 4. These brackets show how ordered pair is intrinsic to definition of function and so a representation of function has embedded in it representation(s) of ordered pairs. A parameter is intrinsic to definition of families of functions so a representation of a family of functions has embedded in it representations of parameter. A note of caution is needed. The embedded brackets do not arise simply because the embedding is possible. To have the overlapping and embedded categories requires that both the super-concept and sub-concept are part of the conversation about the shared representation.

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[2] The CAS-Intensive Mathematics project is a curriculum development and educational research endeavor funded by the National Science Foundation Grant No. TPE 96-18029 to The Pennsylvania State University with a major subcontract to The University of Iowa.

Table 1. The MAGICAL categories

| *MAGICAL* CODE LETTER AND ACT | DESCRIPTION OF THE CATEGORY |
|---|---|
| M<br>MANIPULATE | Change one representation of a mathematical thing to another representation of the same type. This includes standard symbolic manipulation. |
| AU<br>AUGMENT | Make prominent something that is already in a representation. This changes the representation's visual appearance but does not change the essential parts of the representation and it does not change either the type of representation or the mathematical thing. |
| G<br>GENERATE | This is the production or introduction of the first representation of a particular representational type. The new representation may come from an existing representation or from a situation embedded in a (seemingly) real-world context. The actual production may be done via technology, via learner or via teacher.[3] |
| I<br>INTERPRET | This involves giving meaning to a representation by interpreting it in terms of a situation or in terms of an abstract concept. This includes identifying a property of the concept within the representation. |
| C<br>CONNECT/ COMPARE | Connect or compare two representations of the same type. They represent two different mathematical things. This naturally involves Interpret (I) but transcends I by involving two same-type representations of two mathematical things. |
| AS<br>ASCRIBE | Produce the first representation of a situation or of an abstract concept. This includes the basic (initial) generation of a mathematical model. |
| L<br>LINK | For one mathematical thing (equivalence class) a representation of one type is connected to a representation of another type. This link may focus on components of the mathematical thing, thus a linking passage may include a subpassage with a representation of the targeted component. |

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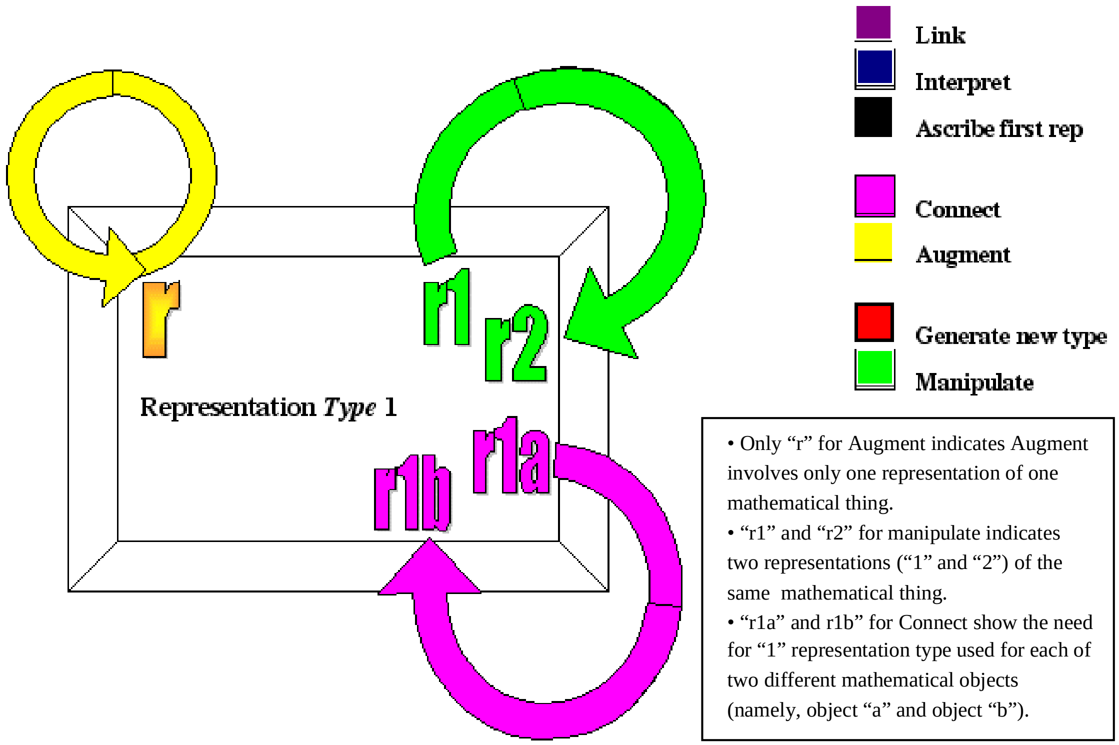
[3] This list likely is not complete. It need not be complete at this point. The basic message is that the student does not have to be the creator and the technology has a potential role in a Generate act.



- Only “r” for Augment indicates Augment involves only one representation of one mathematical thing.
- “r1” and “r2” for manipulate indicates two representations (“1” and “2”) of the same mathematical thing.
- “r1a” and r1b” for Connect show the need for “1” representation type used for each of two different mathematical objects (namely, object “a” and object “b”).

Figure 2. MAGICAL Categories Within One Representation Type

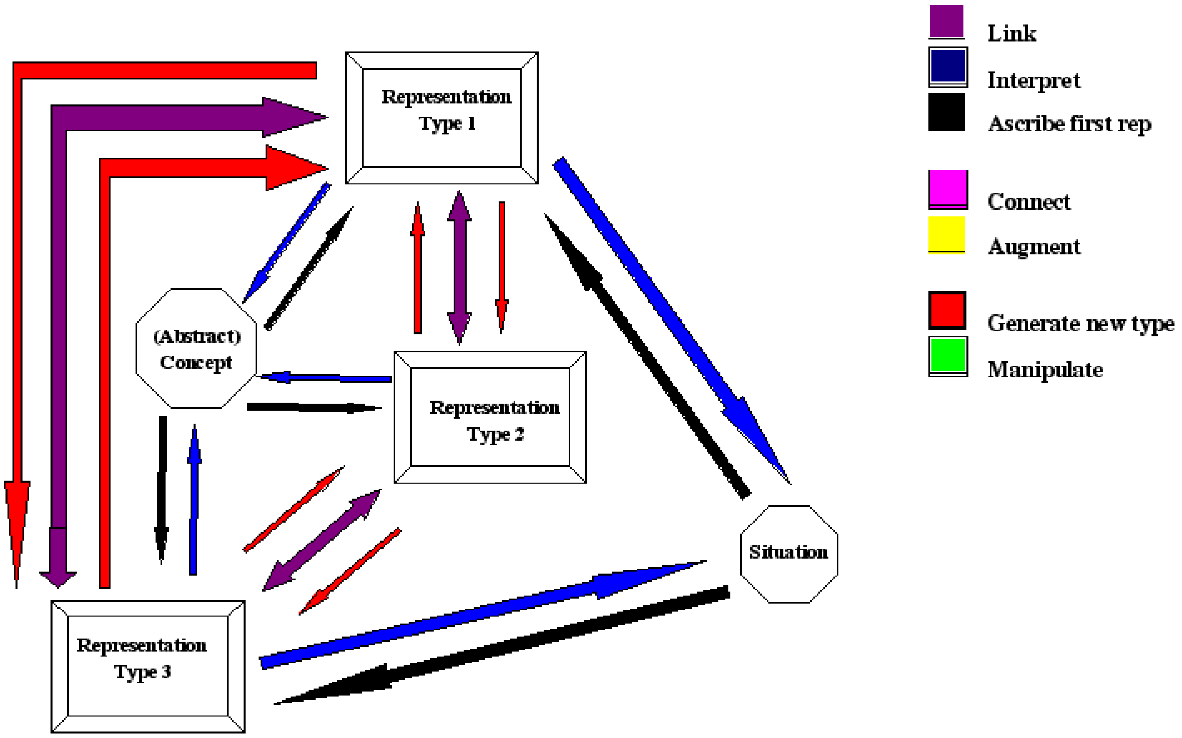


Figure 3. MAGICAL Categories Across Representation Types

(There certainly are more than three representation types, of course, but I am confined by the 2D limits of this page.)

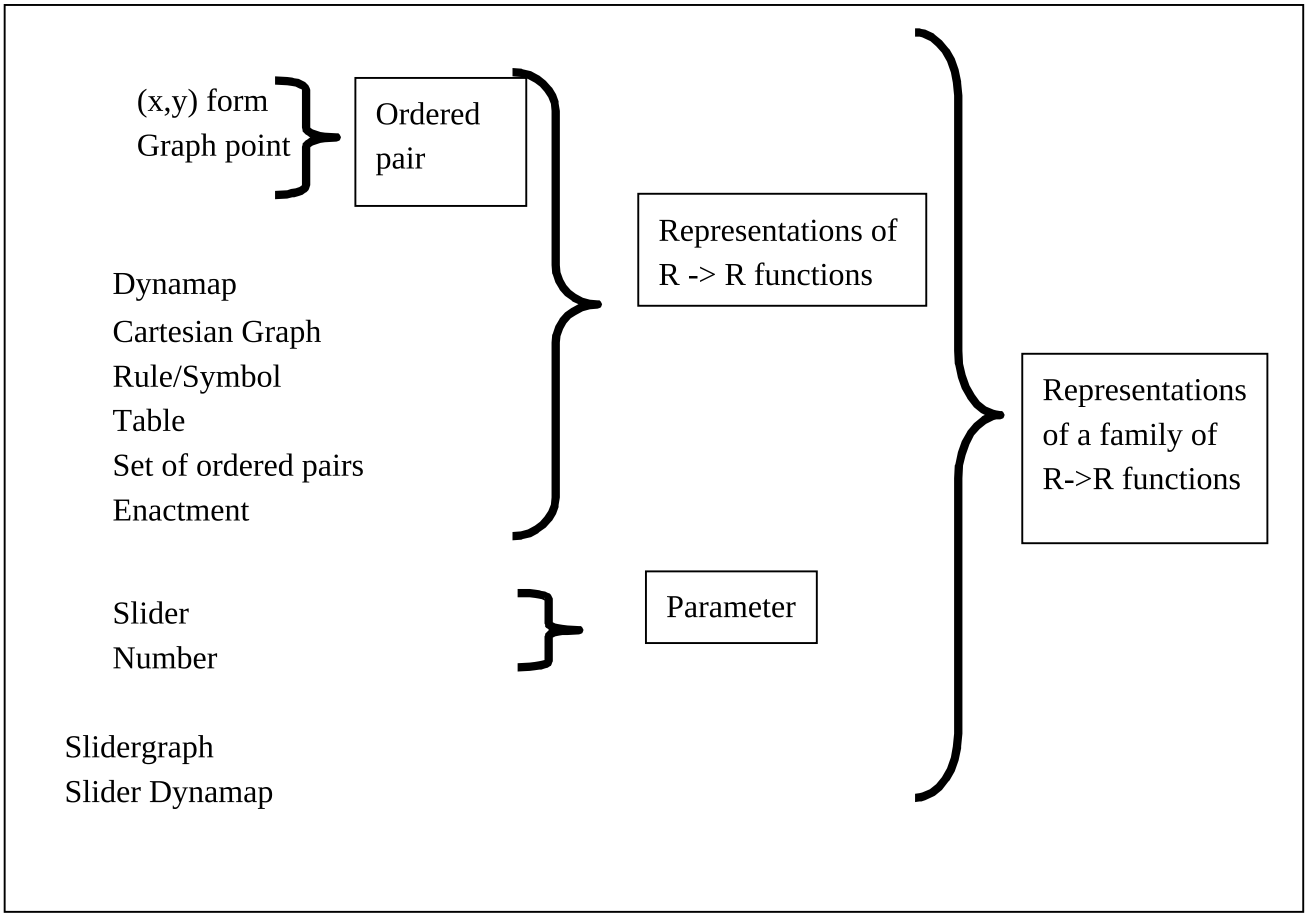


Figure 4. Overlapping nature of representational episodes arising from the nested nature of mathematical things of interest. (List is representative rather than comprehensive)

I can also group the MAGICAL categories by the influence of technology. More about this relationship between representation and technology will appear after a brief discussion of technology features and affordances.

## Technology Features and Affordances

Interest in representations is partially an attempt to explore how affordances of technology relate to students' uses of and actions on representations. There are several *features of technology* that merit consideration as to how they may be viewed from the MAGICAL framework.

I do not consider feature of technology itself to be either a representation or an act with a representation. Certain features however seem to relate closely to particular components of the framework. One example is the TRACE feature in many graphing utilities. This feature seems to be related to Augmenting a representation, as when the TRACE feature is activated and a dot appears to make more prominent a point that the user acknowledges was already on the Cartesian graph. At the same time, the TRACE feature often prints the coordinates of this point on the screen. These coordinates are a numerical representation of the point for which the dot is a geometric representation. Using the TRACE feature in this case also is a Generating act in that it produces the numerical-coordinate representation for the point already represented graphically as the dot. Figure 5 summarizes the TRACE feature relationships to these MAGICAL categories.

| | |
|---|---|
| TRACE: | Au – Add points already in Cartesian graph.<br>G – Produce coordinates for point. |

Figure 5. TRACE feature (in a graphing utility) relates to the MAGICAL categories of Augment and Generate.

A more complex insight comes from looking at the DRAG feature and how it affords various MAGICAL representation acts. DRAG relates to several different MAGICAL scenarios depending in part on when and how as well as in what software the feature is invoked. Figure 6 conveys several ways in which DRAG corresponds to representational actions. It is important to note that DRAG may occur with very different levels of intensity of mathematical thinking about or with the representations.

These variations on MAGICAL categories for a single feature imply several other important points about the ordered triples. First, the command, motion, or other technology feature should not be used as a determining factor for a coding triple.[4] Dragging as a feature associates with several MAGICAL categories. Further, the type of technology and the type of representation used also are not definitive indicators of the MAGICAL codes. The samples in Figure 6 are examples of this inability to use representation type or tool type to determine uniquely a MAGICAL code. For example, the slidergraphs in the DRAG samples in Figure 6 are involved in Augment as well as in Link. Also, dynamaps and slidergraphs can both be created with Cabri Geometry or with The Geometer’s Sketchpad. The correspondence of dynamaps or slidergraphs with Connect, Augment, and Link in Figure 6 indicate clearly that these geometry tools do not uniquely define any one MAGICAL code.

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[4] I also ponder how the MAGICAL categories may help me to explain why CAS does not have the same effects or ease of use or ease of activity generation as do dynamic construction environments. For example, CAS is expert in M as a black box but the Geometer's Sketchpad or Cabri requires more thoughtful action on the user’s part. Augmenting may be harder with CAS than with Sketchpad or Cabri In addition, CAS ease for L via linked multiple representations may underscore the often seemingly excessive use of graphical symbols over literal symbols in many CAS papers, curriculum materials and research reports.

| | |
|---|---|
| M – Drag in Theorist[5] to factor.<br>G – Drag in Fathom[6] to create a scatter plot from data collection. | <- These need not involve deep meaning.<br>Critical mathematical work can be done by the technology. |
| C – Drag vertex point to change figure representing one triangle into a figure representing another triangle[7].<br>Au – Drag x and see (x,f(x)) move in a Cartesian slidergraph.[8] | <- These involve changing representations to underscore meaning or features.<br>Technology can help. |
| I – This category of activity can not be done.<br>As – This category of activity can not be done.<br>L – Compare ordered pairs and motion of points as one drags input point of a dynamap. | <- These involve giving or expressing meaning.<br>Technology may facilitate visualization but it does not help in the critical act of comparing. |

Figure 6. DRAG feature may have one of several relationships to the MAGICAL categories.

## Data to Code

Our research group used the MAGICAL codes primarily with individual student interviews. We also used the MAGICAL categories to analyze portions of the small group work and classroom observations. We similarly used the categories to analyze portions of the written textbook, the answer key, the interview protocol, the small group task, and the teacher interviews. The coding of curriculum materials focused on the content most immediate in both time and content to the time and content of the corresponding interviews, small group work, and classroom observations.

In any of these data arenas, we coded passages that involve not only single students but also passages that include the work of several students and/or adults. Interviewer, researcher, teacher, collaborator, and other adult elements (e.g., questions,

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[5] Theorist refers to the symbol manipulation software MathView.

[6] Fathom is the dynamic statistics software recently released by Key Curriculum Press/Key College Press.

[7] This example refers to dragging in a dynamic construction environment.

[8] This is done in the CAS-IM curriculum and research work. We use *slidergraphs* to represent families of functions. Dragging points along horizontal lines (“sliders”) at the bottom of the screen allows the user to change the values of the family parameters and thus denote over time various members of a family of functions. A Cartesian graph of the family member matching the parameter choices at the given moment appears on the screen and is updated automatically.

statements, tech moves) are included in passages. If students did not follow up on a teacher element, that teacher element was coded as a separate passage.

## Origin of MAGICAL Ideas

Existing expository and empirical literature naturally influenced the MAGICAL framework. “Multiple linked representations” for functions and the question of the role of symbolic manipulation in CAS settings were key starting points. Linking and Manipulating thus arose. Thinking about symbolic manipulation as a means for changing forms but not types of representation led to using types of representations as an essential part of the framework. Work with representations, with early exposure to the “graph as picture” misconception [CITATION], brought out the idea of Interpreting. Thinking about the connection and representation process standards (NCTM, 2000) raised the issue that there is a Linking across different types of representations but there is also a Connecting of representations of the same type. Connecting and Linking were distinguished also because of how differently software tends to support the thinking needed for these representational acts. Modeling literature and the creation of a mathematical model inspired the need for Generating. Creating examples that were not previously part of a mathematical discourse suggested a need for introducing new mathematical objects and thus led to the distinction between Ascribing a first representation of a new mathematical object (as opposed to Generating a representation of a different type for a known mathematical object). Work with tracing features of graphing calculators and other features of the technology as well as the communicative function of representation use contributed to the need to include Augmenting in the framework.

## Relation to Frameworks and Models in the Literature

The resulting MAGICAL framework is consistent with several other writers’ work with algebra, function, and technology. For example, Kieran (1996, 2001) discusses several types of algebraic activity. Two of the three categories seem to have a direct parallel in the MAGICAL framework. Her generating and transformation match Generating and Manipulating, respectively. Aspects of Kieran’s meta category relate in various ways to the categories described here. Noting structure relates to both Linking and Connecting. Kieran’s justifying and proving within the meta category typically involve multiple MAGICAL actions. Modeling relates to Ascribing and Interpreting. One may seek other comparisons between Kieran’s algebraic activities and the MAGICAL representational activities.

The MAGICAL framework also could be used to elaborate on Schneider’s (2000) discussion of representation, operation, and interpretation with CAS. It is also consistent with the reification model used by O'Callaghan (1998) to describe student understanding within a course using Computer-Intensive Algebra. Modeling would be Ascribe; Interpreting is Interpret; Translating involves Generating and Linking; Reifying would be indicative of one who can move within and among the MAGICAL aspects with several types of representation. In addition, students' fluency with some parts of the MAGICAL categories may explain what Pierce (2001) describes as algebraic insight.

The MAGICAL framework transcends all of these in that it transcends algebra, function, and CAS. These categories should apply to any mathematical concept and should be appropriate with any mathematical tool.

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# APPENDIX A.

## Framework Codes with Descriptions and Examples

This reiterates and elaborates on ideas from the MAGICAL framework paper. It includes examples and particular comments, in a form that parallels the TECH USE codes for types and purposes of technology.

| ***Code*** | ***Description*** | ***Examples*** |
|---|---|---|
| M<br>Manipulate | Change one representation of a mathematical thing to another representation of the same type. This includes standard symbolic manipulation. | Solve equation with CAS.<br>Change window for graph by ZOOM or by WINDOW.<br>*Key idea: This changes the form and the information given by the representation.* |
| Au<br>Augment | Make prominent something that is already in a representation. This changes the representation visually but does not change the essential parts of the representation and it does not change either the type of representation or the mathematical thing. | TRACE a graph.<br>Add color, line format, or shading to a geometry figure (as in make the hypotenuse of right triangle red or make the altitudes dotted)<br>*Key idea: This adds no new information but makes existing information more apparent*. |

*Two representations of the same mathematical thing: If new information is added, the code is Manipulate.*
*If there's no new information added, the code is Augment.*

| | | |
|---|---|---|
| G<br>Generate | This is the production or introduction of the first representation of that type. The new representation may come from an existing representation or from a situation embedded in a (seemingly) real-world context. The actual production may be done via technology, via learner or via teacher.[9] | Having the function rule in the CAS, produce the first graph of the function.<br>*Key idea: This requires movement from one representation* TYPE *to another representation* TYPE. |

*Generate is different from Manipulate based on the types of representations involved:*
*Manipulate moves from one representation of the ONE TYPE to another representation of the* SAME *TYPE.*
*Generate moves from one representation of ONE TYPE to a representation of* ANOTHER *TYPE.*

| | | |
|---|---|---|
| I<br>Interpret | This involves giving meaning to a representation by interpreting it in terms of a situation or in terms of an abstract concept. This includes identifying a property of the concept within the representation. | Looking at the constant term of a polynomial function expression, identify the vertical intercept.<br>Using a graph, identify the monotonic nature of a function.<br>*Note: This requires mathematical meaning rather than visual description.* |

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[9] This list likely is not complete. It need not be complete at this point. The basic message is that the student is not the only creator and the technology has a role in this.

| | | |
|---|---|---|
| C<br>Connect/<br>Compare | Connect or compare two representations of the same type. They represent two different mathematical things. This naturally involves I but transcends I by involving two same-type representations of two mathematical things. | Noting floor rounds down and ceiling rounds up while looking at the graphs of both functions on the same graph.<br>Looking at rules $f(x) = x$ and $f(x) = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$, note that the identity function and the absolute value function have the same output for positive numbers.<br>*Key idea: The use of two representations of the same type allows one to compare (find similarities or differences among) two mathematical ideas.* |

*Connect is different from Interpret in that Connect has two different mathematical ideas involved while Interpret involves only one mathematical idea.*

| | | |
|---|---|---|
| As<br>Ascribe | Produce the first representation of a situation or of an abstract concept. This includes the basic generation of a mathematical model. | Construct a rectangle to represent a field for a maximum-area/minimum-perimeter fencing-the-field problem.<br>Fit a function to data points.<br>Having previously fitted a quadratic function to data points now fit a quartic function to those data points.<br>*Key idea: The situation is mathematized in a new way. This includes the creation of a (new) model.* |

*Generate is moving from representation to representation. Ascribe is moving from idea to representation.*

| | | |
|---|---|---|
| L<br>Link | For one mathematical thing (equivalence class) a representation of one type is connected to a representation of another type. This may involve components of the mathematical thing, thus a linking passage may include a subpassage with a representation of the targeted component. | The constant term of a polynomial function expression is related to the vertical intercept.<br>The constant terms in the linear factors in the completely factored form of the denominator of a rational function are related to the vertical asymptotes.<br>*Key idea: The focus is on a mathematical feature of a mathematical entity that is represented in two different forms.* |

*If the representations are of the SAME FORM, the action is Connecting rather than Linking.*