MAT 415 – 001

Spring 2009

Test One

Due on Wednesday, March 4

Do 7 problems.

1. A ring element *a* is called idempotent if $a^2 = a$. Prove that the only idempotents in an integral domain are 0 and 1.

2. *a*, *b* are elements of a commutative ring and *ab* is a zero-divisor. Show that either *a* or *b* is a zero-divisor.

3. $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ is a ring. Prove that $S = \{a + bi \mid a, b \in \mathbb{Z},\ b \text{ is even}\}$ is a subring of $\mathbb{Z}[i]$ but not an ideal of $\mathbb{Z}[i]$.

4. *R* is a ring and *I* is an ideal of *R*. prove that *R*/*I* is commutative if and only if $rs - sr \in I$ for all $r, s \in R$.

5. *R* is a commutative ring and $A \subseteq R$. Show that the annihilator of *A* $\mathrm{Ann}(A) = \{r \in R \mid ra = 0 \text{ for all } a \in A\}$ is an ideal of *R*.

6. Give an example to show that the intersection of two prime ideals need not be a prime ideal.

7. If *R* is an integral domain and *A* is a proper ideal of *R*, must *R*/*A* be an integral domain?

8. Show that the homomorphic image of a PID is a PID.

9. Show that the quotient field of $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ isomorphic to $\mathbb{Q}[i] = \{a + bi \mid a, b \in \mathbb{Q}\}$.

10. In an integral domain, show that *a* and *b* are associates if and only if (*a*) = (*b*).