#1 $$\int_4^\infty \frac{1}{x(\ln x)^3}\,dx$$

$$= \lim_{R\to\infty} \int_4^R \frac{1}{x(\ln x)^3}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int \frac{du}{u^3} = -\tfrac{1}{2}\,{}^1/u^2 + C$$

$$= \lim_{R\to\infty} -\tfrac{1}{2}\frac{1}{(\ln x)^2}\Big|_4^R$$

$$= \lim_{R\to\infty} -\tfrac{1}{2}\frac{1}{(\ln R)^2} + \tfrac{1}{2}\frac{1}{(\ln 4)^2} = \tfrac{1}{2}\frac{1}{(\ln 4)^2}$$

#2 ratio test.

$$\lim_{k\to\infty}\left|\frac{\dfrac{1\cdot3\cdot5\cdots(2k+1)}{2\cdot5\cdot8\cdots(3k+2)}}{\dfrac{1\cdot3\cdot5\cdots(2k-1)}{2\cdot5\cdot8\cdots(3k-1)}}\right| = \lim_{k\to\infty}\frac{1\cdot3\cdot5\cdots(2k+1)}{1\cdot3\cdot5\cdots(2k-1)}\,\frac{2\cdot5\cdot8\cdots(3k-1)}{2\cdot5\cdot8\cdots(3k+2)}$$

$$= \lim_{k\to\infty}\frac{2k+1}{3k+2} = {}^2/_3$$

series converges by the ratio test

limit comparison test

#3

$$\lim_{k\to\infty} \frac{\frac{1}{\sqrt{k^2+2k}}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{\sqrt{k^2+2k}} = 1$$

series diverges by limit comparison to harmonic series

integral test

#4 $\sum_{k=2}^{\infty} \frac{\ln k}{k}$

$f(x) = \frac{\ln x}{x}$ is positive
continuous
decreasing $f'(x) = \frac{1-\ln x}{x^2}$
< 0
if $x > e$

$$\int_2^{\infty} \frac{\ln x}{x}\,dx = \infty$$

series diverges by the integral test

root test

#5

$$\lim_{k\to\infty} \sqrt[k]{\left|\frac{e^{2k}}{k^k}\right|} = \lim_{k\to\infty} \frac{e^2}{k} = 0$$

series converges by the root test

geometric series

#6 $\sum_{k=0}^{\infty} (-1)^k \frac{3^{k+1}}{2^k}$

geometric series $r = -3/2$

$|r| = |-3/2| > 1$

series is divergent geometric series

#7 p-series

$p = 3/4$ diverges

#8 check for absolute convergence

$$\sum_{k=2}^{\infty} \frac{k}{k^3-1}$$

$$\lim_{k\to\infty} \frac{\frac{k}{k^3-1}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^3}{k^3-1} = 1$$

series of absolute values converges by limit comparison to p-series $p=2$

series converges absolutely

#9 check for absolute convergence

$$\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k+1}$$

$$\lim_{k\to\infty} \frac{\frac{\sqrt{k}}{k+1}}{\frac{1}{\sqrt{k}}} = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

series of absolute values diverges by limit comparison to p-series $p = 1/2$

check for conditional convergence

series alternates

$$\frac{\sqrt{k}}{k+1} > \frac{\sqrt{k+1}}{k+2}$$

$$\left(\frac{\sqrt{k}}{k+1}\right)' = -\frac{k-1}{2\sqrt{k}\,(k+1)^2}$$

$$\lim_{k\to\infty} \frac{\sqrt{k}}{k+1} = 0$$

series converges conditionally by the alternating series test

#10 test for divergence

$$\lim_{k\to\infty} \frac{k}{\sqrt{k^2+1}} = 1 \neq 0$$

series diverges by the test for divergence

#11 limit comparison test

$$\lim_{k\to\infty} \frac{\frac{k+1}{k(k+2)}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^2+k}{k^2+2k} = 1$$

series diverges by limit comparison to harmonic series

#12 Ratio test

$$\lim_{k\to\infty}\left|\frac{\frac{(-1)^{k+1}x^{k+1}}{(k+2)(k+3)}}{\frac{(-1)^k x^k}{(k+1)(k+2)}}\right| = \lim_{k\to\infty}|x|\frac{(k+1)(k+2)}{(k+2)(k+3)}$$

$$= |x|$$

diverges $\overset{?}{(}$ converges $\overset{?}{)}$ diverges

-1 0 1

If $x=-1$ $\sum_{k=0}^{\infty}\frac{(-1)^k(-1)^k}{(k+1)(k+2)} = \sum_{k=0}^{\infty}\frac{1}{(k+1)(k+2)}$

converges by comparison to p-series $p=2$

If $x=1$ $\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)(k+2)}$

converges absolutely

#13 ratio test

$$\lim_{k\to\infty} \left| \frac{\frac{(-1)^{k+2}(x-5)^{k+1}}{(k+1)5^{k+1}}}{\frac{(-1)^{k+1}(x-5)^k}{k\,5^k}} \right|$$

$$= \lim_{k\to\infty} |x-5| \frac{k}{k+1} \frac{5^k}{5^{k+1}} = \frac{|x-5|}{5}$$

converges if $\frac{|x-5|}{5} < 1$

$|x-5| < 5$

$-5 < x-5 < 5$

$0 < x < 10$

diverges ? converges ? diverges

0 5 10

if $x=0$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-5)^k}{k\,5^k}$$

$(-1)^{odd} = -1$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{2k+1}}{k}$$ diverges harmonic series

if $x=10$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}5^k}{k\,5^k}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$ converges alternating harmonic series

#14

$$\frac{4}{2+3x} = 4\,\frac{1}{2+3x}$$

$$= \frac{4}{2}\,\frac{1}{1+3x/2}$$

$$= 2\,\frac{1}{1-\left(-\frac{3x}{2}\right)}$$

$\leftarrow u$

$$\frac{4}{2+3x} = 2\left(1 - \frac{3x}{2} + \frac{9x^2}{4} - \frac{27x^3}{8} + \cdots\right)$$

$$= 2\sum_{k=0}^{\infty}\left(-\frac{3x}{2}\right)^k$$

$$= 2\sum_{k=0}^{\infty}(-1)^k\,\frac{3^k x^k}{2^k}$$

$$= \sum_{k=0}^{\infty}(-1)^k\,\frac{3^k x^k}{2^{k-1}}$$

converges for $\left|-\frac{3x}{2}\right| < 1$

$$|-3x| < 2$$

$$-2 < 3x < 2$$

$$-2/3 < x < 2/3$$

diverges $\overset{d}{(}$ converges $\overset{d}{)}$ diverges

$-2/3$ 0 $2/3$

#15 $e^u = 1 + u + u^2/2! + u^3/3! + u^4/4! + \cdots$

$$= \sum_{k=0}^{\infty} \frac{u^k}{k!}$$ converges for all u

$$e^{-t^2} = 1 - t^2 + t^4/2 - t^6/3! + t^8/4! - \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{k!}$$

$$e^{-t^2} - 1 = -t^2 + t^4/2 - t^6/3! + t^8/4! - \cdots$$

$$= \sum_{k=1}^{\infty} (-1)^k \frac{t^{2k}}{k!}$$

$$\int_0^x (e^{-t^2} - 1)\,dt$$

$$= \left(-t^3/3 + t^5/5\cdot 2 - t^7/7\cdot 3! + t^9/9\cdot 4! - \cdots\right)\Big|_0^x$$

$$= -x^3/3 + x^5/5\cdot 2 - x^7/7\cdot 3! + x^9/9\cdot 4!$$

$$= \sum_{k=1}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)k!}\Big|_0^x$$

$$= \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)k!}$$

converges for all x

#16 $\cos u = 1 - u^2/2 + u^4/4! - u^6/6! + u^8/8! - \cdots$

$\cos\sqrt{x} = 1 - x/2 + x^2/4! - x^3/6! + x^4/8! - \cdots$

$$\int_0^{0.5} \cos\sqrt{x}\,dx = \int_0^{0.5} \left(1 - x/2 + x^2/4! - x^3/6! + x^4/8! - \cdots\right)dx$$

$$= \left(x - \frac{x^2}{2\cdot 2} + \frac{x^3}{3\cdot 4!} - \frac{x^4}{4\cdot 6!} + \frac{x^5}{5\cdot 8!} - \cdots\right)\Bigg|_0^{0.5}$$

$$= \underbrace{0.5 - \frac{0.5^2}{2\cdot 2} + \frac{0.5^3}{3\cdot 4!}}_{\text{approximation}} - \frac{0.5^4}{4\cdot 6!} + \frac{0.5^5}{5\cdot 8!} - \cdots$$

less than error

≈ 0.439236