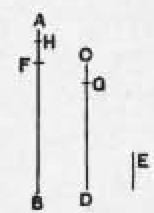
Euclid
The Thirteen Books of The Elements
trans. Sir Thomas L. Heath
Dover

BOOK VII. PROPOSITIONS.

PROPOSITION 1.

Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another.

For, the less of two unequal numbers AB, CD being continually subtracted from the greater, let the number which is left never measure the one before it until an unit is left;



I say that AB, CD are prime to one another, that is, that an unit alone measures AB, CD.

For, if AB, CD are not prime to one another, some number will measure them.

Let a number measure them, and let it be E; let CD, measuring BF, leave FA less than itself,

let AF, measuring DG, leave GC less than itself,

and let GC, measuring FH, leave an unit HA.

Since, then, E measures CD, and CD measures BF,

therefore E also measures BF.

But it also measures the whole BA;

therefore it will also measure the remainder AF.

But AF measures DG;

therefore E also measures DG.

But it also measures the whole DC;

therefore it will also measure the remainder CG.

But CG measures FH;

therefore E also measures FH.

But it also measures the whole FA;

therefore it will also measure the remainder, the unit AH, though it is a number: which is impossible.

Therefore no number will measure the numbers AB, CD; therefore AB, CD are prime to one another. [VII. Def. 12]

Q. E. D.

It is proper to remark here that the representation in Books VII. to IX. of numbers by straight lines is adopted by Heiberg from the MSS. The method of those editors who substitute *points* for lines is open to objection because it practically necessitates, in many cases, the use of specific numbers, which is contrary to Euclid's manner.

"Let CD, measuring BF, leave FA less than itself." This is a neat abbreviation for saying, measure along BA successive lengths equal to CD until a point F is reached such that the length FA remaining is less than CD; in other words, let BF be the largest exact multiple of CD contained in BA.

Euclid's method in this proposition is an application to the particular case of prime numbers of the method of finding the greatest common measure of two numbers not prime to one another, which we shall find in the next proposition. With our notation, the method may be shown thus. Supposing the two numbers to be a, b, we have, say,

$$\begin{array}{l} b)\,a\,(p \\ \quad \underline{pb} \\ \quad c)\,b\,(q \\ \qquad \underline{qc} \\ \qquad d)\,c\,(r \\ \qquad\quad \underline{rd} \\ \qquad\quad 1 \end{array}$$

If now a, b are not prime to one another, they must have a common measure e, where e is some integer, not unity.

And since e measures a, b, it measures $a - pb$, i.e. c.

Again, since e measures b, c, it measures $b - qc$, i.e. d,

and lastly, since e measures c, d, it measures $c - rd$, i.e. 1:

which is impossible.

Therefore there is no integer, except unity, that measures a, b, which are accordingly prime to one another.

Observe that Euclid assumes as an axiom that, if a, b are both divisible by c, so is $a - pb$. In the next proposition he assumes as an axiom that c will in the case supposed divide $a + pb$.

Proposition 2.

Given two numbers not prime to one another, to find their greatest common measure.

Let AB, CD be the two given numbers not prime to one another.

Thus it is required to find the greatest common measure of AB, CD.

If now CD measures AB—and it also measures itself—CD is a common measure of CD, AB.

And it is manifest that it is also the greatest; for no greater number than CD will measure CD.

But, if CD does not measure AB, then, the less of the numbers AB, CD being continually subtracted from the greater, some number will be left which will measure the one before it.

For an unit will not be left; otherwise AB, CD will be prime to one another [VII. 1], which is contrary to the hypothesis.

Therefore some number will be left which will measure the one before it.

Now let CD, measuring BE, leave EA less than itself, let EA, measuring DF, leave FC less than itself, and let CF measure AE.

Since then, CF measures AE, and AE measures DF, therefore CF will also measure DF.

But it also measures itself;

therefore it will also measure the whole CD.

But CD measures BE;

therefore CF also measures BE.

But it also measures EA;

therefore it will also measure the whole BA.

But it also measures CD;

therefore CF measures AB, CD.

Therefore CF is a common measure of AB, CD.

I say next that it is also the greatest.

For, if CF is not the greatest common measure of AB, CD, some number which is greater than CF will measure the numbers AB, CD.

Let such a number measure them, and let it be G.

Now, since G measures CD, while CD measures BE, G also measures BE.

But it also measures the whole BA;

therefore it will also measure the remainder AE.

But AE measures DF;

therefore G will also measure DF.

But it also measures the whole DC;

therefore it will also measure the remainder CF, that is, the greater will measure the less: which is impossible.

Therefore no number which is greater than CF will measure the numbers AB, CD;

therefore CF is the greatest common measure of AB, CD.

Porism. From this it is manifest that, if a number measure two numbers, it will also measure their greatest common measure.

Q. E. D.

Here we have the exact method of finding the greatest common measure given in the text-books of algebra, including the *reductio ad absurdum* proof that the number arrived at is not only a common measure but the *greatest* common measure. The process of finding the greatest common measure is simply shown thus:

$$\begin{array}{l} b)\,a\,(p \\ \underline{pb} \\ \;\;c)\,b\,(q \\ \;\;\underline{qc} \\ \;\;\;\;d)\,c\,(r \\ \;\;\;\;\underline{rd} \end{array}$$

We shall arrive, says Euclid, at some number, say d, which measures the one before it, i.e. such that $c = rd$. Otherwise the process would go on until we arrived at unity. This is impossible because in that case a, b would be prime to one another, which is contrary to the hypothesis.

Next, like the text-books of algebra, he goes on to show that d will be *some* common measure of a, b. For d measures c;
therefore it measures $qc + d$, that is, b,
and hence it measures $pb + c$, that is, a.

Lastly, he proves that d is the *greatest* common measure of a, b as follows.
Suppose that e is a common measure greater than d.
Then e, measuring a, b, must measure $a - pb$, or c.

Similarly e must measure $b-qc$, that is, d: which is impossible, since e is by hypothesis greater than d.

Therefore etc.

Euclid's proposition is thus *identical* with the algebraical proposition as generally given, e.g. in Todhunter's algebra, except that of course Euclid's numbers are integers.

Nicomachus gives the same rule (though without proving it) when he shows how to determine whether two given *odd* numbers are prime or not prime to one another, and, if they are not prime to one another, what is their common measure. We are, he says, to compare the numbers in turn by continually taking the less from the greater as many times as possible, then taking the remainder as many times as possible from the less of the original numbers, and so on; this process "will finish either at an unit or at some one and the same number," by which it is implied that the division of a greater number by a less is done by *separate subtractions* of the less. Thus, with regard to 21 and 49, Nicomachus says, "I subtract the less from the greater; 28 is left; then again I subtract from this the same 21 (for this is possible); 7 is left; I subtract this from 21, 14 is left; from which I again subtract 7 (for this is possible); 7 will be left, but 7 cannot be subtracted from 7." The last phrase is curious, but the meaning of it is obvious enough, as also the meaning of the phrase about ending "at one and the same number."

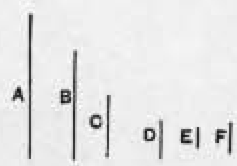
The proof of the Porism is of course contained in that part of the proposition which proves that G, a common measure different from CF, must measure CF. The supposition, thereby proved to be false, that G is greater than CF does not affect the validity of the proof that G measures CF in any case.

Proposition 3.

Given three numbers not prime to one another, to find their greatest common measure.

Let A, B, C be the three given numbers not prime to one another;

thus it is required to find the greatest common measure of A, B, C.



For let the greatest common measure, D, of the two numbers A, B be taken; [VII. 2]

then D either measures, or does not measure, C.

First, let it measure it.

But it measures A, B also;

therefore D measures A, B, C;

therefore D is a common measure of A, B, C.

I say that it is also the greatest.

For, if D is not the greatest common measure of A, B, C, some number which is greater than D will measure the numbers A, B, C.

Let such a number measure them, and let it be E.

Since then E measures A, B, C,

it will also measure A, B;

therefore it will also measure the greatest common measure of A, B. [VII. 2, Por.]

But the greatest common measure of A, B is D;

therefore E measures D, the greater the less: which is impossible.

Therefore no number which is greater than D will measure the numbers A, B, C;

therefore D is the greatest common measure of A, B, C.

Next, let D not measure C;

I say first that C, D are not prime to one another.

For, since A, B, C are not prime to one another, some number will measure them.

Now that which measures A, B, C will also measure A, B, and will measure D, the greatest common measure of A, B. [VII. 2, Por.]

But it measures C also;

therefore some number will measure the numbers D, C;

therefore D, C are not prime to one another.

Let then their greatest common measure E be taken. [VII. 2]

Then, since E measures D,

and D measures A, B,

therefore E also measures A, B.

But it measures C also;

therefore E measures A, B, C;

therefore E is a common measure of A, B, C.

I say next that it is also the greatest.

For, if E is not the greatest common measure of A, B, C, some number which is greater than E will measure the numbers A, B, C.

Let such a number measure them, and let it be F.

Now, since F measures A, B, C,
it also measures A, B;
therefore it will also measure the greatest common measure of A, B. [VII. 2, Por.]

But the greatest common measure of A, B is D;
therefore F measures D.

And it measures C also;
therefore F measures D, C;
therefore it will also measure the greatest common measure of D, C. [VII. 2, Por.]

But the greatest common measure of D, C is E;
therefore F measures E, the greater the less: which is impossible.

Therefore no number which is greater than E will measure the numbers A, B, C;
therefore E is the greatest common measure of A, B, C.

Q. E. D.

Euclid's proof is here longer than we should make it because he distinguishes two cases, the simpler of which is really included in the other.

Having taken the greatest common measure, say d, of a, b, two of the three given numbers a, b, c, he distinguishes the cases

(1) in which d measures c,

(2) in which d does not measure c.

In the first case the greatest common measure of d, c is d itself; in the second case it has to be found by a repetition of the process of VII. 2. In either case the greatest common measure of a, b, c is the greatest common measure of d, c.

But, after disposing of the simpler case, Euclid thinks it necessary to prove that, if d does not measure c, d and c must necessarily *have* a greatest common measure. This he does by means of the original hypothesis that a, b, c are not prime to one another. Since they are not prime to one another, they must have a common measure; any common measure of a, b is a measure of d, and therefore any common measure of a, b, c is a common measure of d, c; hence d, c must have a common measure, and are therefore not prime to one another.

The proofs of cases (1) and (2) repeat exactly the same argument as we saw in VII. 2, and it is proved separately for d in case (1) and e in case (2), where e is the greatest common measure of d, c,

(α) that it is a common measure of a, b, c,

(β) that it is the *greatest* common measure.

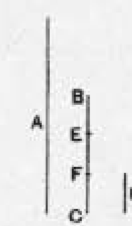
Heron remarks (an-Nairizi, ed. Curtze, p. 191) that the method does not only enable us to find the greatest common measure of *three* numbers; it can be used to find the greatest common measure of as many numbers as we please. This is because any number measuring two numbers also measures their greatest common measure; and hence we can find the G.C.M. of pairs, then the G.C.M. of pairs of these, and so on, until only two numbers are left and we find the G.C.M. of these. Euclid tacitly assumes this extension in VII. 33, where he takes the greatest common measure of *as many numbers as we please*.

PROPOSITION 4.

Any number is either a part or parts of any number, the less of the greater.

Let A, BC be two numbers, and let BC be the less;
I say that BC is either a part, or parts, of A.

For A, BC are either prime to one another or not.



First, let A, BC be prime to one another.

Then, if BC be divided into the units in it, each unit of those in BC will be some part of A;
so that BC is parts of A.

Next let A, BC not be prime to one another;
then BC either measures, or does not measure, A.

If now BC measures A, BC is a part of A.

But, if not, let the greatest common measure D of A, BC be taken; [VII. 2]
and let BC be divided into the numbers equal to D, namely BE, EF, FC.

Now, since D measures A, D is a part of A.

But D is equal to each of the numbers BE, EF, FC;
therefore each of the numbers BE, EF, FC is also a part of A;
so that BC is parts of A.

Therefore etc.

Q. E. D.

The meaning of the enunciation is of course that, if a, b be two numbers of which b is the less, then b is either a *submultiple* or *some proper fraction* of a.

(1) If a, b are prime to one another, divide each into its units; then b contains b of the same parts of which a contains a. Therefore b is "parts" or a *proper fraction* of a.

(2) If a, b be not prime to one another, either b measures a, in which case b is a submultiple or "part" of a, or, if g be the greatest common measure of a, b, we may put $a = mg$ and $b = ng$, and b will contain n of the same parts (g) of which a contains m, so that b is again "parts," or a *proper fraction*, of a.