* #1 $\displaystyle\int_2^\infty \frac{1}{(x+3)^4}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{(x+3)^4}\,dx$

let $u = x+3$ $\quad \int u^{-4}\,du = -\frac{1}{3u^3} + C$

$du = dx$

$\displaystyle\lim_{t\to\infty} -\frac{1}{3(x+3)^3}\Big|_2^t = \lim_{t\to\infty}\left(-\frac{1}{3(t+3)^3}\right) + \frac{1}{3(5)^3}$

$\displaystyle = \frac{1}{375}$

#2 $\displaystyle \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243} + \cdots$

$2/3 \quad 2/3 \quad 2/3$

geometric series $\quad a = 4/9$

$r = 2/3$ converges

$\displaystyle \text{sum} = \frac{4/9}{1 - 2/3} = 4/3$

#3 $\displaystyle\sum_{k=1}^{\infty} \left(\frac{3}{4k}\right)^k$

root test

$$\lim_{k\to\infty} \sqrt[k]{\left(\frac{3}{4k}\right)^k} = \lim_{k\to\infty} \frac{3}{4k} = 0$$

series converges by root test

#4 $\displaystyle\sum_{k=2}^{\infty} \frac{k}{\sqrt{k^5+5}}$

$$\frac{k}{\sqrt{k^5+5}} < \frac{k}{\sqrt{k^5}} = \frac{k}{k^{5/2}} = \frac{1}{k^{3/2}}$$

$$\sum_{k=2}^{\infty} \frac{k}{\sqrt{k^5+5}} < \sum_{k=2}^{\infty} \frac{1}{k^{3/2}}$$

p-series
$p = 3/2$ converges

series converges by comparison
to p-series $p = 3/2$

#5 $\sum_{k=1}^{\infty} \frac{1}{(2k+1)(\ln(2k+1))^2}$

consider $f(x) = \frac{1}{(2x+1)(\ln(2x+1))^2}$

positive, continuous, decreasing

$\int_1^{\infty} \frac{1}{(2x+1)(\ln(2x+1))^2} = \frac{1}{2\ln 3}$ integral converges

series converges by the integral test

#6 $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k+\ln k}}$

$$\frac{1}{k\sqrt{k+\ln k}} < \frac{1}{k\sqrt{k}} = \frac{1}{k^{3/2}}$$

$$\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k+\ln k}} < \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$$

p-series
$p = 3/2$ converges

series converges by comparison
to p-series $p = 3/2$

#7 $\sum_{k=1}^{\infty} \frac{k^4}{k!}$

$$\lim_{k\to\infty} \left| \frac{\frac{(k+1)^4}{(k+1)!}}{\frac{k^4}{k!}} \right| = \lim_{k\to\infty} \frac{(k+1)^4}{k^4} \frac{k!}{(k+1)!}$$

$$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^{4 \to 1} \frac{1}{k+1}^{\to 0} = 0$$

series converges by ratio test

#8 $\sum_{k=2}^{\infty} \frac{k^2+1}{k^{3.5}-2}$

$$\lim_{k\to\infty} \frac{\frac{k^2+1}{k^{3.5}-2}}{\frac{1}{k^{1.5}}} = \lim_{k\to\infty} \frac{k^{3.5}+k^{1.5}}{k^{3.5}-2}$$

$= 1$

series converges by limit comparison to p-series $p=1.5$

*9 $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1.1}\ln(k+1)}$

consider series of absolute values

$$\sum_{k=1}^{\infty} \frac{1}{k^{1.1}\ln(k+1)} < \sum_{k=1}^{\infty} \frac{1}{k^{1.1}}$$

series of absolute values converges by comparison to p-series $p=1.1$

series converges absolutely

#10 $-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\cdots+(-1)^k\frac{k}{k+1}+\cdots$

$$\lim_{k\to\infty} \frac{k}{k+1} = 1 \neq 0$$

series diverges by the test for divergence

#11 $\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{k^3+1}$

Consider series of absolute values

$\sum_{k=1}^{\infty} \frac{k^2}{k^3+1}$

$\lim_{k\to\infty} \frac{\frac{k^2}{k^3+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^3}{k^3+1} = 1$

series of absolute values diverges by limit comparison to harmonic series – not absolutely convergent

alternates

$\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{k^3+1}$

$\left(\frac{k^2}{k^3+1}\right)' = -\frac{(k^3-2)}{(k^3+1)^2} < 0 \text{ if } k>1$

decreasing

$\lim_{k\to\infty} \frac{k^2}{k^3+1} = 0$

series converges conditionally by AST

*12 $$\sum_{n=1}^{\infty} \frac{2^n (x+3)^n}{3n}$$

$$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1}(x+3)^{n+1}}{3(n+1)}}{\frac{2^n (x+3)^n}{3n}} \right| = \lim_{n\to\infty} \frac{3n}{3(n+1)} \frac{2^{n+1}}{2^n} |x+3|$$

$$= 2|x+3|$$

converges if $2|x+3| < 1$

$$|x+3| < 1/2$$

$$-1/2 < x+3 < 1/2$$

$$-7/2 < x < -5/2$$

If $x = -5/2$

$$\sum_{n=1}^{\infty} \frac{2^n (1/2)^n}{3n} = 1/3 \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges}$$

If $x = -7/2$

$$\sum_{n=1}^{\infty} \frac{2^n (-1/2)^n}{3n} = 1/3 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{converges}$$

converges $-7/2 \le x < -5/2$

#13

$$\frac{1}{2+x^2} = \frac{1}{2}\,\frac{1}{1+x^2/2} = \frac{1}{2}\,\frac{1}{1-(-x^2/2)}$$

$u = -x^2/2$

$$\frac{1}{2+x^2} = \tfrac{1}{2}\left(1 - x^2/2 + x^4/4 - x^6/8 + x^8/16 - \cdots\right)$$

$$-1 < -x^2/2 < 1$$
$$1 > x^2/2 > -1$$
$$-2 < x^2 < 2$$
$$-\sqrt{2} < x < \sqrt{2}$$

$$= \tfrac{1}{2}\sum_{n=0}^{\infty}\left(-x^2/2\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^{n+1}}$$

#14

$$1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots$$

$$= 1 + x^3 + \frac{(x^3)^2}{2!} + \frac{(x^3)^3}{3!} + \frac{(x^3)^4}{4!} + \cdots$$

$$= 1 + u + u^2/2! + u^3/3! + u^4/4! + \cdots$$

where $u = x^3$

e^u

Function is e^{x^3}

* #15 $\int_0^1 \sin(x^2)\,dx$

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots \qquad -\infty < u < \infty$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

$$\int_0^1 \sin(x^2)\,dx = \left(\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots\right)\Bigg|_0^1$$

$$= \underbrace{\frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!}}_{\text{approximation}} - \frac{1}{15\cdot 7!} + \cdots$$

$$\approx 0.31028$$